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A user's guide: Relative Thom spectra via operadic Kan extensions

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1. Key insights and central organizing principles

The main idea of [**Bea17**] is to give a "Third Isomorphism Theorem" for quotients of ring spectra by actions of n-fold loop spaces. Recall the classical Third Isomorphism Theorem:

Theorem 1 (Noether). *Let $N \subseteq K \subseteq G$ be a composition of inclusions of normal subgroups. Then there is an isomorphism of groups:*

$$G/K \cong (G/N)/(K/N).$$

We can generalize this theorem to any action of a Lie group on a (real or complex) smooth manifold:

Theorem 2 (Bourbaki). *Let G be a Lie group acting freely, properly and smoothly on a finite dimensional smooth X. Let H be a normal Lie subgroup of G. Then the canonical projection map $X \to X/H$ defines, upon taking quotients by G, an isomorphism of smooth manifolds:*

$$X/G \cong (X/H)/(G/H).$$

In general, we see that when one has an action of a group G on an object X, and a normal subgroup $H \subset G$, it is reasonable to expect that X/G and $(X/H)/(G/H)$ will be very similar to each other.

In [**ABG$^+$14**] it was shown that Thom spectra (e.g. the classical cobordism spectra) can be described as quotients of ring spectra by the actions of n-fold loop spaces. Thus the question arises: given an n-fold loop space G acting on a ring spectrum R, and "normal subgroup" $H \subseteq G$, what is the relationship between R/G and $(R/H)/(G/H)$? Of course to even answer this question one must make rigorous a number of notions. Namely, subgroups of n-fold loop spaces, actions of n-fold loop spaces on ring spectra and quotients of ring spectra by such actions. It's also going to be important that n-fold loop spaces are algebras for the little

n-cubes operad, and as such admit monoid structures up to coherent homotopy (where the level of coherence increases with n).

1.1. "Normal subgroups" of n-fold loop spaces. Given a discrete group G with a subgroup H, the most important thing one gets from normality of H is the ability to equip the quotient set G/H with a group structure. One might even say that this is the defining property of being normal; this is the property that makes normality worth knowing about. In the case of an n-fold loop space, we might similarly ask for H to have some property such that G/H is still an n-fold loop space. Completely characterizing such sub-objects is very hard, but we are rescued by a preponderance of natural examples of this structure.

Our analogy for the inclusion of a normal subgroup $H \subseteq G$ will be a fiber sequence of n-fold loop spaces $F \to E \to B$, where the maps are maps *of* n-fold loop spaces. In this analogy, F will play the role of H, E the role of G, and B the role of G/H. Recall that for a morphism of discrete groups $\phi\colon G \to K$ we can take the kernel, the subgroup of G that goes to 1 under ϕ. Categorically, this is equivalent to taking the pullback of the cospan $0 \to G \leftarrow K$. In the case that K is G/H for some normal subgroup, we have that the kernel of ϕ, the pullback of that cospan, is isomorphic to H. Thus replacing G and K with n-fold loop spaces, and replacing that pullback with a homotopically coherent pullback (e.g. taking the pullback in the quasicategory of n-fold loop spaces), we generalize the notion of normal subgroup to n-fold loop spaces.

Key Idea 1.1. *When working with n-fold loop spaces instead of strict groups, we should replace the notion of "H is a normal subgroup of G and $G/H \cong K$" with "there is a homotopy fiber sequence of n-fold loop spaces $H \to G \to K$."*

Probably the most obvious examples of such structure are just the loop space functor applied n times to any fiber sequence of simplicial sets. For instance, we might take the well known Hopf-fibration $S^1 \to S^3 \to S^2$, and take some number of based loops on that fibration. Thus we would have that $\Omega^n S^1$ is a "normal sub-n-fold loop space" of $\Omega^n S^3$, and $\Omega^n S^3/\Omega^n S^1 = \Omega^n S^2$. Recall that we also have a number of compact Lie groups that are infinite loop spaces, and there are many infinite loop maps going between them. Thus we get fiber sequences of infinite loop spaces like $U \to O \to O/U$, $SU \to U \to S^1$. We will see several other such fiber sequences in later sections when we discuss examples.

1.2. Actions of n-fold loop spaces on ring spectra. To discuss actions of n-fold loop spaces on ring spectra, we need to decide on a notion of what we mean by ring spectra. Because it was the notion used in [**Bea17**], we'll work with the symmetric monoidal quasicategory of ring spectra described in [**Lur14**]. Ultimately the choice of model of ring spectra seems inessential, but there might be difficulties that this author isn't aware of. We will also assume that our ring spectra admit multiplicative structures parameterized by *operads* (more precisely, the ∞-operads of [**Lur14**]), in particular, the little n-disk operads, $\mathbb{E}_n$, of Boardman and Vogt [**BV73**]. It would be a distraction to describe operads in detail here, but suffice it to say that $\mathbb{E}_1$ things should be thought of as *homotopy associative with all higher coherence data*, $\mathbb{E}_\infty$ should be thought of as *homotopy associative and commutative with all higher coherence data*, and $\mathbb{E}_n$ should be thought of as *homotopy associative with all higher coherence data and homotopy commutative with some higher coherence data*, where "some" will increase as n increases. There is a close connection between n-fold loop spaces and $\mathbb{E}_n$-ring spectra: in particular the suspension spectrum of an n-fold loop space is always an $\mathbb{E}_n$-ring spectrum.

Generally, to define an action of a monoid G on an object X, one provides a map of monoids: $f : G \to Aut(X)$, and that's exactly what we'll do here. Specifically, we know from [**ABG15**] (as well as plenty of other places) that if R is an $\mathbb{E}_n$-ring spectrum then there is an n-fold loop space of homotopy automorphisms of R as an $\mathbb{E}_n$-ring, which we'll denote by $GL_1(R)$. So we'll say that an n-fold loop space G acts on an $\mathbb{E}_n$-ring spectrum R if there is a morphism of n-fold loop spaces $f\colon G \to GL_1(R)$. We will often want to go between the map $f\colon G \to GL_1(R)$ and $Bf\colon BG \to BGL_1(R)$ by looping and delooping, so we will always assume that our n-fold loop spaces are connected.

1.3. Quotients of ring spectra by actions of n-fold loop spaces. Let's suppose we have an $\mathbb{E}_n$-ring spectrum R and a map of $(n-1)$-fold loop spaces $f\colon BG \to BGL_1(R)$, hence an action of an n-fold loop space G on R. By thinking of left R-modules as a quasicategory $LMod_R$, we can think of $BGL_1(R)$ as a sub-quasicategory of Mod_R in the following way: the base point of $BGL_1(R)$ is R itself, paths in $BGL_1(R)$ are the obvious homotopy automorphisms of R, higher cells in $BGL_1(R)$ are mapped to the necessary higher degree morphisms in $LMod_R$. In other words there is an inclusion of simplicial sets $BGL_1(R) \hookrightarrow LMod_R$ which precisely picks out the action of $GL_1(R)$ on R as a left R-module. Thus a morphism of $(n-1)$-fold loop spaces $Bf\colon BG \to BGL_1(R)$ can be thought of as picking out a BG-shaped diagram inside of $LMod_R$ describing an action of G on R by left R-module automorphisms.

To construct the "quotient" of R by this action in a homotopically invariant way, we should imagine forcing all of the morphisms in the image of BG to be equivalent to the identity morphism. One might imagine manually attaching 2-cells between each morphism and the identity morphism, and then attempting to make this coherent. Unfortunately this would require giving an infinite list of coherences, and coherences of coherences, and coherences of coherences of... ad infinitum. Luckily, the theory of [**Lur09**] and [**Lur14**] builds all coherence data into the category theory itself, making this completely formal: we will say that the quotient of R by the G action is the *quasicategorical* colimit (i.e. a quasicategorical analog of the classical notion of a homotopy colimit, hence homotopy invariant), in $LMod_R$, of the morphism $BG \to BGL_1(R) \hookrightarrow LMod_R$. Our intuition about classical categorical colimits is the right one in this case and R/G, or $colim(BG \to BGL_1(R) \hookrightarrow LMod_R)$, is precisely the universal left R-module on which G acts homotopically trivially.

Key Idea 1.2. *An action of an n-fold loop space G on an $\mathbb{E}_n$-ring spectrum R is the data of a functor $f : BG \to LMod_R$ that takes the unique basepoint of BG to R and every path in BG to an $\mathbb{E}_n$-algebra automorphism of R. The quotient of R by such an action is the R-module spectrum colimit(f).*

Now that we have all of the relevant notions in place, we can state the main theorem of [**Bea17**]:

Theorem. *Suppose $Y \xrightarrow{i} X \xrightarrow{q} B$ is a fiber sequence of reduced $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$ with i and q both maps of $\mathbb{E}_n$-algebras. Let $f \colon X \to BGL_1(\mathbb{S})$ be a morphism of $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$. Then there is a a morphism of $\mathbb{E}_{n-1}$-algebras $B \to BGL_1(M(f \circ i))$ whose associated Thom spectrum is equivalent to Mf.*

Note that in the actual statement we have to be more precise then we've been up to this point. In particular, instead of talking about n-fold loop spaces, we specifically talk about Kan complexes which are algebras for the $\mathbb{E}_n$-operad. We also assume that our Kan complexes are reduced. This means that as simplicial sets they have a single zero simplex. This guarantees, for instance, that they are connected, but it also makes things technically simpler. In the end, any time one has a connected Kan complex, one can replace it up to homotopy with a reduced one. Note also that in the statement of the theorem we write Mf and $M(f \circ i)$ instead of $\mathbb{S}/\Omega Y$ or $\mathbb{S}/\Omega X$. This is purely a notational convention and is done stay aligned with classical descriptions of such quotient spectra as Thom spectra.

1.4. Important examples of quotients and iterated quotients. It turns out that there are many examples in homotopy theory of taking quotients of ring spectra by n-fold loop space actions. As described above, these are typically called Thom spectra. Recall that there is a map of infinite loop spaces $j \colon BO \to BGL_1(\mathbb{S})$ called the j-homomorphism. So any time we have a map of infinite loop spaces (or just of n-fold loop spaces, since we can just forget some structure) $BG \to BO$, we can take the induced quotient $\mathbb{S}/G$. Thus we can obtain all of the classical Thom spectra in this way: MO, MSO, MU, MSU, MSp, $MSpin$ and so forth. However, it is also known that one can produce the Eilenberg-MacLane spectra $H\mathbb{Z}$, $H\mathbb{Z}/p$ and $H\widehat{\mathbb{Z}}_p$ in this way (though sometimes these require taking quotients of the p-local sphere $\mathbb{S}_{(p)}$ rather than just $\mathbb{S}$). These last three examples are originally due to Mahowald and Hopkins but a modern treatment can be found in [**AJB08**]. In particular, there is map $f : \Omega^2 S^3 \to BGL_1(\mathbb{S}_{(p)})$ whose associated quotient is $H\mathbb{Z}/p$ and there is a map $g : \Omega^2 S^3\langle 3\rangle \to BGL_1(\mathbb{S}_{(p)})$ which is factored by f and has associated quotient $H\widehat{\mathbb{Z}}_p$.

For *iterated* quotients, we can first notice that all of the classical cobordism spectra described above come from infinite loop spaces maps $BG \to BO$ and that most of these factor through BSO. So we can think of BG as being the fiber of a map $BSO \to B(SO/G)$. Thus we get that $\mathbb{S}/O \simeq (\mathbb{S}/G)/(SO/G)$. For technical reasons it will help to have all of our groups connected, so we'll always work with SO instead of O. There are ways to get around this problem, but it wouldn't be productive to spend time on them in this user's guide. As an example, recall that there is a fiber sequence of infinite loop spaces $BSpin \to BSO \to B\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 2)$. Thus we have that $\mathbb{S}/SO = MSO \simeq (\mathbb{S}/Spin)/(SO/Spin) \simeq MSpin/\mathbb{R}P^\infty$.

We get another interesting example by considering the fiber sequence of 2-fold loop spaces $\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \to \mathbb{C}P^\infty \simeq S^1$ along with the map $f : \Omega^2 S^4 \to BGL_1(\mathbb{S}_{(p)}$ described in [**AJB08**]. In this way, we see that $H\mathbb{Z}/p \simeq H\widehat{\mathbb{Z}}_p/\Sigma^\infty_+\mathbb{Z}$ where the $\Sigma^\infty_+\mathbb{Z}$ action is of course given by multiplication by p. One especially nice benefit of this description is that the above construction is obtained as an equivalence of $\mathbb{E}_1$-ring spectra.

Finally, near and dear to the heart of this author, is the example of the sequence of spectra $X(n)$ defined by Ravenel in [**Rav86**] and used to prove the Nilpotence Theorem in [**DHS88**]. Each $X(n)$ is obtained from a map of 2-fold loop spaces $\Omega SU(n) \to BGL_1(\mathbb{S})$, i.e. $X(n) = \mathbb{S}/\Omega^2 SU(n)$. Recalling that there are fiber sequences $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$, we have that $X(n+1) \simeq X(n)/\Omega^2 S^{2n+1}$.

1.5. So what? One really interesting consequence of all this comes from the fact that whenever we have a k-fold loop map $BG \to BGL_1(R)$ for R an $\mathbb{E}_n$ ring spectrum (with $n > k$), we get a so-called Thom isomorphism of spectra $R/G \wedge_R R/G \simeq R \wedge_{\mathbb{S}} \Sigma^\infty BG_+$. So for the examples above we obtain equivalences of spectra like: $H\mathbb{Z}/p \wedge_{H\widehat{\mathbb{Z}}_p} H\mathbb{Z}/p \simeq H\mathbb{Z}/p \wedge_{\mathbb{S}} S^1$ and $MSpin \wedge_{MSO} MSpin \simeq MSpin \wedge_{\mathbb{S}} \Sigma^\infty_+ K(\mathbb{Z}/2, 2)$. Thus the methods described in [**Bea17**] give relative Künneth theorems for many well known Thom spectra.

Additionally, the main theorem above can be thought of as one direction of a generalized Galois correspondence. In particular, recall that in a Galois extension $E \to F$, intermediate Galois extensions are in bijection with normal subgroups of $Gal(E/F)$. In our case, we're showing that if we have a fiber sequence of n-fold loop spaces $H \to G \to G/H$ and a map of $(n-1)$-fold loop spaces $BG \to BGL_1(R)$, then we get a composition of spectra $R \to R/H \to R/G$. It is not true that we can recover R/H from R/G or R from R/H by taking homotopy fixed points, but we can often perform these recoveries by taking so-called homotopy *co*fixed points. Thus this composition is not an iterated Galois extension but an iterated Hopf-Galois extension in the sense of [**Rog08**]. So our main theorem gives a distinctly algebro-geometric interpretation to many classical (and geometric!) morphisms of cobordism spectra.

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