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A user's guide: Relative Thom spectra via operadic Kan extensions

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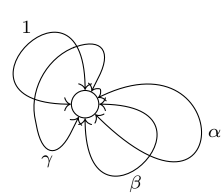
2. Metaphors and imagery

Presenting Thom spectra as quotients of ring spectra by actions of $\mathbb{E}_n$-spaces, as in [**And14**], ends up meaning that the most important metaphors for us involve thinking about groups acting on things. I'll explain below how I think about groups (or $\mathbb{E}_n$-spaces) acting on spectra, how I think about taking quotients by these actions, and finally how I think about quasicategories in general.

2.1. Group actions as functors. Given an $\mathbb{E}_n$-monoidal space G, we can form its classifying spaces BG, which is an $\mathbb{E}_{n-1}$-monoid and has the property that $\Omega BG \simeq G$. There are a number of ways to construct BG, including the well-known bar construction, but these are not always intuitively enlightening. However, since $\Omega BG \simeq G$, we at least have a good way to think about $\pi_1(BG)$. That is to say, a *path* in BG that starts and ends at the base point corresponds to multiplication by a point in G. Given a group (or $\mathbb{E}_n$-space) G with points $\{1, \alpha, \beta, \gamma\}$, I think of BG as something like the following picture:

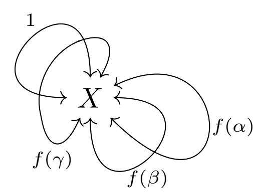


(1)

In other words, BG looks like a single connected component with paths attached to it for each point of G. Note that BG is a space (or simplicial set) and has higher homotopy groups, so the above picture does not show all of the structure of BG. For example, if G had a relation like $\alpha\gamma^{-1} = \beta$, one would have to glue in a 2-cell to BG to manifest this relation.

If we think of BG as a category (or a quasicategory) it would have a single object and a 1-morphism for each point of G. Importantly, since we can write $G \simeq \Omega BG$, we know that every point in G admits an "inverse" corresponding to traversing the loop in the opposite direction. Thus since multiplication by a point in G is always invertible (up to homotopy), BG, thought of as a category, is actually a groupoid (or ∞-groupoid).

Suppose now that we have an $\mathbb{E}_{n-1}$-monoidal quasicategory $\mathcal{C}$ and an $\mathbb{E}_{n-1}$-monoidal functor $f : BG \to \mathcal{C}$. Since BG only has one object $*$, the functor f must pick out a diagram in $\mathcal{C}$ that looks like a single object and a collection of equivalences of that object (as well as higher coherent cells). Moreover, since the functor f is $\mathbb{E}_{n-1}$-monoidal it must take $*$ to an $\mathbb{E}_n$-algebra in $\mathcal{C}$ and each loop in BG to a morphism of $\mathbb{E}_{n-1}$-algebras. It also has to take 1 to the identity morphism (or at least a morphism equivalent to the identity), which will be relevant in understanding the colimit of this diagram. Graphically, if $f(*) = X$, then the image of f looks like a diagram in $\mathcal{C}$ of the following form:

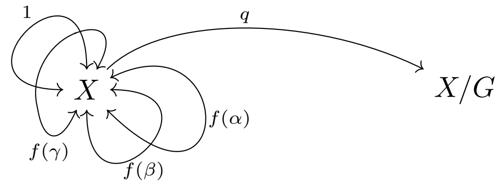


(2)

So the functor $f : BG \to \mathcal{C}$ can be though of as picking out an *action* of G on X. Given a point in G, f tells us exactly how to use it to transform X by choosing an equivalence of X in $\mathcal{C}$ associated to it. What's also really great about this formalism is that by asking for f to be a functor of quasicategories, it retains all the (potentially very complicated) coherence data contained in G.

2.2. Quotients of actions as colimits. Now we can elaborate on what it means to take the "quotient" of a G-action when the G action is described as above. There are two competing pictures here that I use, depending on how I'm thinking about this quotient. We touched on both of them in Topic 1, but we'll flesh them out further here.

2.2.1. *A categorical description.* The first one is the simplest (and the most "correct"), but also perhaps less intuitive. If we return to picture (2) above, and remember the idea behind the colimit of a diagram in $\mathcal{C}$, we know that $colim(f(BG))$ should be a single object of C that admits a morphism *from* the diagram $f(BG)$. So we have a picture like this:



(3)

But the really important thing about the fact that X/G is the colimit of this diagram is that this new diagram, shown in (3) above, has to commute. In other words, all possible concatenations of loops in picture (2) followed by the morphism $q\colon X \to X/G$ have to be equivalent. And moreover, any *other* object admitting a morphism from picture (2) that commutes with all those loops has to admit a map from X/G factoring that morphism. In particular, for any two loops $f(\alpha)$ and $f(\beta)$ in $f(BG)$ we have to have that $q \circ f(\alpha) \simeq q \circ f(\beta)$. So, in particular, $q \circ f(\alpha) \simeq q \circ 1$. Thus X/G really is the universal thing admitting a map from X on which the G-action is forced to be homotopically trivial.

2.2.2. *A topological description.* The second way of thinking about this quotient is a bit more "hand-wavy." Since all of this is happening in the setting of quasicategories, which are a special type of simplicial set, we can use some of our intuition about working with simplicial sets. In particular, we can think about *attaching cells* to make the loops in $f(BG)$ homotopically trivial.

If we were working with a *set* X with an action by a group G, we would take the orbits of G-action on X, or the quotient of X by G, by describing the quotient as being the things in X after x has been forced to be *equal* to gx for every $x \in X$ and $g \in G$. When doing homotopy theory, and especially higher category theory, it's considered *evil* to ask for things to actually be equal. Instead, if we want two things to be "the same" in some way, we put a cell between them. This is why, for instance, if we have a based space Y and take the pushout of $* \leftarrow Y \to *$, we get $*$, but if we take the *homotopy pushout* we get ΣY. The reduced suspension of Y is precisely two points with every way of identifying them being given its own 1-cell $* \leftrightarrow *$.

For our case, we have autoequivalences $f(\alpha)\colon X \to X$ that we want to trivialize. One way to do this would be to "glue" cells to X for each x and gx. But since we're working in quasicategories (i.e. ∞-categories), we would then have to glue in 2-cells between certain 1-cells, and n-cells in general, according to the structure of G. Thus while thinking of the procedure as being "gluing in cells" we're forced to take the abstract colimit described above.

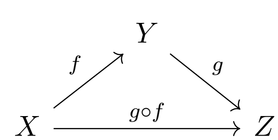
2.3. Twisting and untwisting. Despite the fact that we have focused so far on the idea of taking quotients of group actions, there is also a strong undercurrent of bundle theory, and twisted bundles, throughout [**Bea17**]. Recall that for any group G and a space X, a map $X \to BG$ always determines a principal G-bundle over X. Thus in our case, a map $BG \to BGL_1(\mathbb{S})$ defines a principal $GL_1(\mathbb{S})$-bundle on BG. Note that if we replaced $\mathbb{S}$ with a commutative ring R, we'd equivalently have a bundle of R-modules over BG and this bundle would locally be isomorphic to R. Since it doesn't make sense to talk about a bundle of *spectra* on BG (for instance, what would the total space be?), we can't say the same here, but we still *think* of a map $BG \to BGL_1(\mathbb{S})$ as defining a "bundle of 1-dimensional $\mathbb{S}$-modules over BG."

Suppose now that the map of interest is the trivial one, $*\colon BG \to BGL_1(\mathbb{S})$. Then the induced quotient, $\mathbb{S}/G$ where G acts trivially, is in fact equivalent to $\mathbb{S} \wedge \Sigma^\infty_+ BG = \mathbb{S}[BG]$, the spherical group ring on BG. Compare this with the classical fact that if X is a G-space with a trivial G-action then the Borel construction $X \times_G EG \simeq X \times BG$. But now notice that if, on the other hand, we pulled back a "bundle of sphere spectra" along the trivial map $*\colon BG \to BGL_1(\mathbb{S})$, we should get something that *looks like* $BG \times \mathbb{S}$. Of course that last expression doesn't make sense since BG and $\mathbb{S}$ aren't even in the same category, but this is still a useful perspective.

For a non-trivial map, we can still use this perspective, but we should think of the map $f\colon BG \to BGL_1(\mathbb{S})$ as *twisting* the trivial bundle on BG. And then we should think of the colimit of $BG \xrightarrow{f} BGL_1(\mathbb{S}) \to \mathit{Spectra}$ as being a *twisted tensor product* of BG with $\mathbb{S}$. This construction should be thought of as being analogous to the twisted tensor products that arise when one attempts to compute the homology groups of fibered spaces, as in [**Bro59**].

2.4. Quasicategories. Throughout [**Bea17**] and in many of the references given therein, standard category theory has been replaced by the theory of quasicategories, a model for so-called ∞-categories. Though there isn't space to truly explicate what makes the theory of quasicategories go, I thought it might help to at least provide a picture here of what's going on, and why one might choose to use quasicategories instead of model categories.

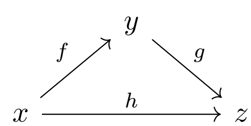
Central to the entire notion of category theory is the idea that there are morphisms between objects and that one can compose a morphism f with a morphism g if the domain of g is equal to the domain of f. When we do that, we get a commutative diagram like the following:



Though it seems obvious, notice that $f \circ g$ is a *new* morphism whose existence we've demanded (otherwise we don't have a category). Moreover, we've demanded that the application of $g \circ f$ to X is *equal* to the application of f followed by the application of g.

For homotopy theorists, asking for something to be equal is a real problem. The main issue is that we might have two categories that are Quillen equivalent (hence have the same "homotopy theory") and things that are equal in one might be only *equivalent* in the other. Thus is general it's too restrictive to demand that compositions be equal. A good example to have in mind here is when f and g are continuous morphisms of spaces. We could ask that $g \circ f(x) = g(f(x))$ for every $x \in X$, or we could simply ask that there is a continuous family of continuous maps $H(t)$ such that $H(0)(x) = g \circ f(x)$ and $H(1)(x) = g(f(x))$.

The way we get around this problem, which is intrinsic to category theory itself, is to replace categories altogether with simplicial sets. Just like categories, simplicial sets still have objects (0-simplices) and morphisms (1-simplices), but they also have a lot of "higher data" that we think of as n-morphisms for $n > 1$. Notice that in an arbitrary simplicial set, it really doesn't make sense to "compose" morphisms. The 1-cells in a simplicial set X are just elements in some set, specifically they're elements of $X([1])$. So to make simplicial sets behave like categories we instead do the following: given two 1-simplices f and g such that the right endpoint of f is the same 0-simplex as the left endpoint of g, we can ask that there be another 1-simplex h going from the left endpoint of f to the right endpoint of g. In other words, we can ask for a triangle in our simplicial set:



So we have that h is modeling the *composition* of g with f. We need one more condition however: we ask that there is a 2-simplex whose boundary is the above triangle. What this is supposed to be modeling is that going along h is "the same" as going along f and then along g. But it's not actually *equal.* That two cell is taking the place of the family of continuous maps from our example.

Visually, I think of the 2-simplex described above as giving me a way of sliding h up to the edge of the triangle made by f and g. And the fact that I'm able to do this is telling me that while h may not be equal to $g(f)$, we at least have a coherent way of wiggling the data of h to the data f and g. If I can find an h and a filling 2-simplex every time I have two 1-simplices that match up the way f and g do, then my simplicial set is behaving a lot like a category. I can always produce a "composition" of morphisms, and this composition can always at least be homotoped back to the things whose composition it represents.

Finally, note that simplicial sets have n-simplices for all n. So asking for a simplicial set to actually be a quasicategory is asking that we can perform that above procedure in all dimensions. In other words, whenever I have 3 composable 2-simplices (think of these as forming a tetrahedron with no bottom side), I can find another 2-simplex that closes the tetrahedron and then, crucially, I can fill in that whole tetrahedron with a 3-simplex. If we can do this *every time* we have $n+1$ composable n-simplices then our simplicial set deserves to be called a quasicategory or ∞-category.

References

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