Enchiridion: Mathematical User's Guides Volume 3 (September 2017)
mathusersguides.com

A user's guide: The Adams-Novikov E_2-term for Behrens' spectrum $Q(2)$ at the prime 3

Don Larson

1. Key insights and central organizing principles

The goal of [**Lar15**] is to compute the E_2-term of a spectral sequence that converges to the homotopy groups of a spectrum called $Q(2)$. This E_2-term, which I denote $E_2^{*,*}Q(2)$, is tied to the stable homotopy groups of spheres π_*S. The latter are known more briefly as the stable stems. You could say there are "six degrees of separation" between $E_2^{*,*}Q(2)$ and the stable stems, because:

(1) $E_2^{*,*}Q(2)$ approximates $\pi_*Q(2)$ via the spectral sequence,
(2) $\pi_*Q(2)$ is the source of a homomorphism to $\pi_*L_{K(2)}Q(2)$ (where $L_{K(2)}$ is localization at the 2nd Morava K-theory at the prime 3),
(3) $\pi_*L_{K(2)}Q(2)$ is in a l.e.s. of homotopy groups with $\pi_*L_{K(2)}S$ arising from a cofiber sequence,
(4) $\pi_*L_{K(2)}S$ is the target of a homomorphism from π_*L_2S arising from a homotopy pullback square (where L_2 is localization at the 2nd Johnson-Wilson theory $E(2)$ at the prime 3),
(5) π_*L_2S is a second-order approximation to $\pi_*S \otimes \mathbb{Z}_3$ by the Chromatic Convergence Theorem of Hopkins and Ravenel, and
(6) $\pi_*S \otimes \mathbb{Z}_3$ approximates π_*S just as looking at the 3-torsion of any group gives you some information about that group.

So, my computation is in the same mathematical neighborhood as π_*S, if only at the outskirts, and that's an exciting place to live! Even more exciting is the surprising role that number theory plays in this computation and in the study of the stable stems in general. One of the bridges connecting number theory and homotopy theory is a spectrum known as topological modular forms, which happens to be a key ingredient in the construction of $Q(2)$. As a result, the organizational structure of my computation is governed completely by elliptic curves, modular forms, and certain maneuvers one can make with them. One such maneuver underlying the construction of $Q(2)$ is a degree 2 isogeny between elliptic curves (i.e., a surjective morphism with a kernel of size 2), which is why the "2" appears in the notation for $Q(2)$. By contrast, the "2" in $K(2)$ and $E(2)$ refers to chromatic level, an idea to be explored later in this user's guide.

Although the notation does not indicate it, the spectrum $Q(2)$ is, for our purposes, linked specifically to the prime 3 as the above list indicates. However, it is one of an infinite family of "Q-spectra" $\{Q(\ell)\}$ obtained by letting the prime p and the isogeny degree ℓ vary. These spectra were constructed by Mark Behrens [**Beh06**] in order to shed light on the p-torsion of the stable stems at all different primes p. We shall see that different things are known about a given $Q(\ell)$ depending on the values of p and ℓ.

1.1. From the Steenrod algebra to BP-theory and elliptic curves.

The starting point of my computation is a Hopf algebroid, a certain algebraic structure. The classical Adams spectral sequence starts with an object of a similar algebraic nature, so let's begin there.

Let p be a prime and let A_* denote the mod p dual Steenrod algebra. The pair $(\mathbb{F}_p, A_*)$ forms a Hopf algebra over $\mathbb{F}_p$. One interpretation of this Hopf algebra structure is to say that, for an $\mathbb{F}_p$-algebra T, the sets $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p, T)$ and $\mathrm{Hom}_{\mathbb{F}_p}(A_*, T)$ are the objects and morphisms, respectively, of a category. The well-known coproduct $\Delta : A_* \to A_* \otimes_{\mathbb{F}_p} A_*$ corresponds to composition of morphisms. Mind you, the category in this case is not exciting, for it only has one object! But it turns out that all morphisms are invertible. In other words, the category is really just a *group* with the obvious map $\eta : \mathbb{F}_p \to A_*$ (called the unit map) corresponding to the group's unit element. On the category level, η "picks out" from a given morphism the single object that acts as both its source and target. The Hopf algebra cohomology of $(\mathbb{F}_p, A_*)$ is precisely the cohomology of the cobar complex

$$\mathbb{F}_p \xrightarrow{\eta} A_* \to A_* \otimes A_* \to A_* \otimes A_* \otimes A_* \to \cdots$$

constituting the E_2-term for the Adams spectral sequence converging to $\pi_* S \otimes \mathbb{Z}_p$.

Passing from the classical Adams spectral sequence to a tool involving an exotic cohomology theory, such as the Adams-Novikov spectral sequence (ANSS) based on BP-theory [**Rav86**], means changing the input format from a Hopf algebra to something slightly more general. The pair of $\mathbb{Z}_{(p)}$-algebras (BP_*, BP_*BP) has the property that, given a $\mathbb{Z}_{(p)}$-algebra T, the sets $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(BP_*, T)$ and $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(BP_*BP, T)$ are the objects and morphisms of a category. All morphisms are once again invertible. This time, though, there is the potential for more than one object, which means the category is a *groupoid*. So the pair is called a *Hopf algebroid* rather than a Hopf algebra. Appropriate, no? Since in this case one can "pick out" either the source or target of a morphism as they are generally different, there are two distinct structure maps $\eta_R, \eta_L : BP_* \to BP_*BP$, the right and left units. The algebra BP_*BP is a left BP_*-module via η_L and a right BP_*-module via η_R. The Hopf algebroid cohomology of (BP_*, BP_*BP) is precisely the cohomology of the cobar complex

$$BP_* \xrightarrow{\eta_R - \eta_L} BP_*BP \to BP_*BP \otimes BP_*BP \to \cdots$$

that constitutes the E_2-term for the ANSS converging to $\pi_*S \otimes \mathbb{Z}_{(p)}$. For an arbitrary spectrum X in place of the sphere S, the ANSS input would be $BP_*(X)$ instead of $BP_*(S) \cong BP_*$, and the target would be $\pi_*X \otimes \mathbb{Z}_{(p)}$.

The spectrum of topological modular forms TMF is an object (depending on p) whose homotopy groups π_*TMF can be interpreted collectively as a refinement of modular forms over $\mathbb{Z}_{(p)}$, hence the name. We shall now see why this interpretation is reasonable.

For the remainder of this section let's fix $p = 3$, as that is the setting for my computation. The ANSS for $X = TMF$ rather than $X = S$ has as its E_2-term the cohomology of a Hopf algebroid over $\mathbb{Z}_{(3)}$ that I denote (B, Γ), where

$$B = \mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}]/(\Delta = q_4^2(16q_2^2 - 64q_4)),$$
$$\Gamma = B[r]/(r^3 + q_2r^2 + q_4r).$$

For fixed values of q_2 and q_4 the equation $y^2 = 4x(x^2 + q_2x + q_4)$ is a Weierstrass equation for a non-singular elliptic curve, and for a fixed r the map $x \mapsto x + r$ is an elliptic curve isomorphism that preserves this particular Weierstrass form [**Sil09**]. Therefore, the groupoid obtained by mapping into a $\mathbb{Z}_{(3)}$-algebra T has certain elliptic curves over T as its objects, and isomorphisms $x \mapsto x + r$ as its morphisms.

Modular forms enter the picture because B itself *is* the ring of modular forms over $\mathbb{Z}_{(3)}$ with respect to the congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}(2, \mathbb{Z})$ [**DS05**]. Moreover, the ANSS E_2-term for TMF is the cohomology of the cobar complex

(1) $$B \xrightarrow{\eta_R - \eta_L} \Gamma \to \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma \otimes \Gamma \to \cdots$$

and a classical computation shows that the 0th cohomology (i.e., $\ker(\eta_R - \eta_L)$) has the form

$$MF = \mathbb{Z}_{(3)}[c_4, c_6, \Delta, \Delta^{-1}]/(1728\Delta = c_4^3 - c_6^2)$$

which is precisely the ring of modular forms over $\mathbb{Z}_{(3)}$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. By the time one arrives at π_*TMF one sees many of these modular forms in one guise or another (e.g., there is a homotopy class corresponding to a multiple of Δ) plus other stuff. The complete computation of π_*TMF is due originally to Hopkins and Miller and is beautifully described in expository works by Tilman Bauer [**Bau08**], Akhil Mathew [**Mat**], and Andre Henriques [**Hen**].

The spectrum TMF has several close cousins, one of which has as its underlying Hopf algebroid the pair (B, B) where $\eta_R = \eta_L = 1$ (an example of a "trivial" Hopf algebroid). In this case each elliptic curve C from the groupoid comes with the additional datum of a choice of order 2 subgroup H, and no morphism $x \mapsto x + r$ can preserve this choice. The ANSS collapses at E_2 and the homotopy groups are described by the graded algebra B itself (the grading of an element determines its degree in homotopy). This spectrum is denoted $TMF_0(2)$.

KEY IDEA 1.1. *The spectra TMF and $TMF_0(2)$ are E_∞-ring spectra, so they receive a map from S and are therefore tied to $\pi_* S$ via the induced homomorphism on homotopy. Moreover, their homotopy groups are fully computable precisely because the algebras constituting their underlying Hopf algebroids are finitely generated, as opposed to $(\mathbb{F}_p, A_*)$ and (BP_*, BP_*BP) whose full cohomologies (let alone $\pi_* S \otimes \mathbb{Z}_p$) will be forever out of reach.*

1.2. Constructing $Q(2)$.

To tie the construction of $Q(2)$ to something familiar, imagine an ordered 2-simplex with vertices $V = \{v_0, v_1, v_2\}$ as shown in Figure 1. Let E denote the set of its edges and F denote the singleton set consisting of its one and only two-dimensional face f. [For technical reasons, E and F actually contain additional phantom elements that are safely ignored—see the second paragraph after this one.] There are two collections of "face" maps

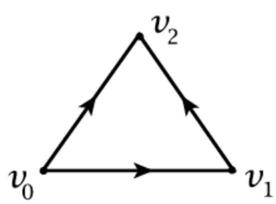

FIGURE 1. An ordered 2-simplex.

attached to this structure: the first is a triple $d_0, d_1, d_2 : F \to E$ where $d_i(f)$ is the edge opposite v_i; the second is a pair $d_0, d_1 : E \to V$ (a blatant abuse of notation) where d_i sends an edge to the vertex opposite its i-th one. Face maps can be composed and they satisfy identities that are straightforward but tedious to write down. We can sum this up with a diagram

$$V \leftleftarrows E \Lleftarrow F \tag{2}$$

displaying the two collections of face maps.

Now imagine applying the free abelian group functor Free$(-)$ to (2), in which case one might denote Free(d_i) by δ_i. Because the maps δ_i are group homomorphisms, they can be added or subtracted as long as they share the same source and target. For example, taking a cue from basic homology, one might take alternating sums of the δ_i, yielding

$$\text{Free}(V) \Leftarrow \text{Free}(E) \Lleftarrow \text{Free}(F) \tag{3}$$

where the doubly thick arrow $\Leftarrow$ represents $\delta_0 - \delta_1$ and the triply thick arrow $\Lleftarrow$ represents $\delta^0 - \delta^1 + \delta^2$.

Diagrams (2) and (3) are each examples of a *simplicial set* (the latter having the additional structure of a simplicial *free abelian group*) that I hope demonstrate that such objects are natural. They happen to contain interesting stuff only in dimensions 0, 1, and 2. In general, simplicial sets can have interesting higher-dimensional stuff, and they also (always) have "degeneracy" maps that go in the direction opposite to the face maps. One of the degeneracy maps in the 2-simplex would send the vertex v_0 to the "degenerate" edge between v_0 and itself, for example. [This degenerate edge at v_0 is one of the aforementioned phantom elements.] However, this aspect of a simplicial structure is not relevant to my computation.

The construction of $Q(2)$ starts with a simplicial *stack*

$$\mathcal{M}_0 \Leftarrow \mathcal{M}_1 \Lleftarrow \mathcal{M}_2 \tag{4}$$

whose structure mirrors (2), but with sets replaced by stacks, hence its name. [A *moduli stack*, or sometimes just *stack*, is an algebraic construction useful for studying parametrized families of mathematical objects.] The $\mathcal{M}_i$ are moduli stacks of elliptic curves and the face maps between them are morphisms of stacks (see below for more on the genesis of these morphisms). The doubly- and triply-thick arrows once again denote alternating sums of the face maps. The precise name for (4) is "semi-simplicial stack," because there are no degeneracy maps in the data, and that is what the prefix "semi-" indicates in this context.

The semi-simplicial stack (4) can be realized topologically. This means there is a functor (contravariant, so arrows get turned around) that takes the diagram of stacks above, to a diagram of spectra. The result is a semi-cosimplicial spectrum

$$TMF \Rightarrow TMF_0(2) \vee TMF \Rrightarrow TMF_0(2). \tag{5}$$

made from the spectra TMF and $TMF_0(2)$ introduced in the previous subsection. This time, $\Rightarrow$ and $\Rrightarrow$ denote alternating sums of coface maps, and the prefix "semi-" indicates that there are no codegeneracy maps in the data.

The spectrum $Q(2)$ is the homotopy limit of the diagram (5).

1.3. The ANSS for $Q(2)$. Recall that the ANSS inputs for TMF and $TMF_0(2)$ are the cohomologies of their respective Hopf algebroids (B, Γ) and (B, B). The key organizing principle of my paper is the ability to pass covariantly from (5) (or contravariantly from (4)) to a diagram of underlying Hopf algebroids

$$(B, \Gamma) \Rightarrow (B, B) \oplus (B, \Gamma) \Rrightarrow (B, B) \tag{6}$$

where $\Rightarrow$ and $\Rrightarrow$ are alternating sums of Hopf algebroid maps.

Key Idea 1.2. *Each Hopf algebroid map in* (6) *encodes a certain maneuver one can make with elliptic curves and their Weierstrass equations, and is algebraically straightforward.*

For example, one of the two maps $TMF \to TMF \vee TMF_0(2)$ is built from a map $TMF \to TMF_0(2)$ which is underlain by a map of Hopf algebroids

$$\phi_f : (B, \Gamma) \to (B, B)$$

that corresponds to "forgetting" the choice of order 2 subgroup within an elliptic curve. The map ϕ_f is, in turn, induced by the algebra map $\Gamma \to B$ that is the identity on B and sends r to 0.

Given (5) and (6), it is plausible that $E_2^{*,*}Q(2)$ can be assembled from the ANSS E_2-terms for TMF and $TMF_0(2)$, and that is indeed the case. That allows me to make my way from (6) to $E_2^{*,*}Q(2)$ by first replacing each Hopf algebroid in (6) by its cobar complex and multiplying the cobar complex differentials d for (B,Γ) by -1 in the middle column. This yields a double complex

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle 0\oplus(-d)} & & \uparrow{\scriptstyle 0} \\
\Gamma\otimes\Gamma & \longrightarrow & 0\oplus\Gamma\otimes\Gamma & \longrightarrow & 0 \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle 0\oplus(-d)} & & \uparrow{\scriptstyle 0} \\
\Gamma & \longrightarrow & 0\oplus\Gamma & \longrightarrow & 0 \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle 0\oplus(-d)} & & \uparrow{\scriptstyle 0} \\
B & \longrightarrow & B\oplus B & \longrightarrow & B
\end{array} \tag{7}
$$

where the horizontal maps are induced by the corresponding alternating sums of Hopf algebroid maps in (6). The E_2-term $E_2^{*,*}Q(2)$ is the cohomology of the totalization of (7).

1.4. Computing the ANSS for $Q(2)$. The construction from the previous subsection implies that $E_2^{*,*}Q(2)$ is computable via the double complex spectral sequence (DCSS) for (7). The first step in this DCSS is to take cohomology with respect to the vertical arrows. If Ext^n denotes the n-th cohomology of (B,Γ) (i.e., the n-th cohomology of the cobar complex (1)) by (which, by homological algebra, is a sensible abbreviation) then, since $\mathrm{Ext}^0 \cong MF$, this yields

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Ext}^2 & \longrightarrow & \mathrm{Ext}^2 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Ext}^1 & \longrightarrow & \mathrm{Ext}^1 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
MF & \longrightarrow & B\oplus MF & \longrightarrow & B
\end{array} \tag{8}
$$

The next step in the DCSS is to take cohomology with respect to the horizontal arrows in (8). Fortunately, much of this work is trivial.

KEY IDEA 1.3. *The groups* Ext^n *are entirely 3-torsion for* $n \geq 1$*, and the horizontal arrows in* (8) *are induced by Hopf algebroid maps that send polynomial algebra generators to multiples of 3, hence identically zero. Therefore, the only non-trivial computation in this step of the DCSS is the cohomology of the 0th row*

$$MF \to B \oplus MF \to B. \tag{9}$$

Because there are only three columns of data in these diagrams, the DCSS has only one additional step past measuring cohomology horizontally, after which the DCSS stabilizes. This additional step requires that I compute the kernel and the image of just a single map (i.e., DCSS differential) whose source is Ext^1 and whose target is the cokernel of the map $B \oplus MF \to B$ from (9).

With a few isolated exceptions, I completely compute the DCSS and obtain a presentation for $E_2^{*,*}Q(2)$ in my paper. The answer is a jungle of 3-torsion that I will not reproduce here (see the main theorem of [**Lar15**]).

1.5. Why $Q(2)$? One key property of $Q(2)$ is the existence of a cofiber sequence

$$L_{K(2)}DQ(2) \to L_{K(2)}S \to L_{K(2)}Q(2) \tag{10}$$

with the $K(2)$-localization of $Q(2)$ on the far right and the $K(2)$-localization of the Spanier-Whitehead dual of $Q(2)$ on the far left. This is a big reason why $\pi_* Q(2)$ is worth pursuing. For other values of p and ℓ, analogous cofiber sequences involving $Q(\ell)$ are only conjectured to exist.

The cofiber sequence (10) is not only a reason to study $Q(2)$, it is also a reason why $Q(2)$ was built in the first place. It turns out that (10) (along with its conjectured analogs for other p and ℓ) is inspired by a pair of earlier results. The first is a result of Adams/Baird and Ravenel that gives, at the prime 2 for example, a cofiber sequence

$$L_{K(1)}S \to KO_2 \to KO_2 \tag{11}$$

(here KO_2 is 2-adic real K-theory). The second is a result of Goerss, Henn, Mahowald, and Rezk, at the prime 3 [**GHMR05**]; namely, there exists a diagram

$$L_{K(2)}S \to X_1 \to X_2 \to X_3 \to X_4 \to X_5 \tag{12}$$

that induces a tower of cofiber sequences with $L_{K(2)}S$ at the top. Behrens' construction of $Q(2)$ as a semi-cosimplicial object is a reinterpretation of the machinery underlying (12) that is more in the spirit of (11) and more explicitly number-theoretic. For example, TMF appears in Goerss-Henn-Mahowald-Rezk but is not identified outright as a number-theoretic object. It appears there in the guise of a certain homotopy fixed point spectrum.

References

[Mat] Akhil Mathew, *The homotopy groups of TMF*. Available at `http://www.math.harvard.edu/~amathew/tmfhomotopy.pdf`.

[Bau08] Tilman Bauer, *Computation of the homotopy of the spectrum* tmf, Groups, homotopy and configuration spaces, 2008, pp. 11–40.

[Beh06] Mark Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), no. 2, 343–402.

[Beh] ———, *Calculation of the rational homotopy of $Q(2)$*. Available at http://www3.nd.edu/~mbehren1/other/LOQ.dvi.

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005.

[Hen] André Henriques, *The homotopy groups of tmf and of its localizations*. Available at http://math.mit.edu/conferences/talbot/2007/tmfproc/Chapter16/TmfHomotopy.pdf.

[GHMR05] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *A resolution of the $K(2)$-local sphere at the prime 3*, Ann. of Math. (2) **162** (2005), no. 2, 777–822.

[Lar15] Donald M. Larson, *The Adams-Novikov E_2-term for Behrens' spectrum $Q(2)$ at the prime 3*, J. Pure Appl. Algebra **219** (2015), no. 10, 4681–4707.

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986.

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, Second, Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009.

Department of Mathematics, Catholic University, Washington, DC 20064

E-mail address: larsond@cua.edu