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A user's guide: The Adams-Novikov E_2-term for Behrens' spectrum $Q(2)$ at the prime 3

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2. Metaphors and imagery

In the mathematical neighborhood of π_*S, I like to imagine the elements of the stable stems all living in a tall building at the center of town. Each lives happily in its own apartment. There must be infinitely many units, then, so let's imagine infinitely many floors (I said tall, right?) and an infinitely long hallway on each floor. In particular, just like Hilbert's eponymous hotel, this building always has vacancies! From that you might surmise it's a warm and welcoming place, where you can walk in and meet some of the tenants, get to know them, find out what they're about, and maybe make some friends. You would not be totally wrong. Walk into the lobby on floor 0 and you find everyone is genuine and forthcoming and glad to tell you about themselves. Someone invites you in for a delicious dinner. Mmmmm. But higher up, things start to change. Folks on floor 1 invite you in for wine and cheese, which is great, but they all have dinner plans with some guy named Adams so you have to leave after the cheese. When you're seen roaming the hall on floor 2 some tenants kindly offer you directions because you look lost. Directions...to the exit! Burn! No cheese for you, unless you happen to be a high-powered homotopy theorist who used to hang out at Princeton in the 1970s. On floor 3 the air is stuffier. The residents pay you no mind because they are kind of a big deal; after all, several of them were featured in a New York Times article back in 1976 [**NYT**]. Undaunted, but perhaps a little hungry, you press on to floor 4. The elevator door opens and a huge bouncer blocks your way. You crane your neck hoping merely for a glimpse of the hallway, but no dice. You ask who lives on this floor: he says nobody. You ask how to get to the higher floors: he says there are no higher floors, which is a lie. He then says "you were never here" and next thing you know you wake up back in the lobby, dazed and confused.

That escalated quickly. Floor 3 is indeed exclusive, and floors ≥ 4 are basically secret societies. Do not despair, however. It's not you. For all practical purposes, *nobody* from the outside has ever been allowed on the upper floors, not even the distinguished topologists who commissioned the building's construction in the first place. What happens up there is in large part a mystery.

2.1. Wine? Cheese? In this apartment building analogy, the tenants (a.k.a. elements of π_*S) on floor n are those detected by $K(n)$-local sphere but not by the $K(n-1)$-local sphere, across all primes p. It is morally true, but not quite precise, to say these are exactly the elements of π_*S represented by cohomology classes on the n-line of the ANSS E_2-page for the sphere across all primes p. And as algebraic topologists push their way forward through each $K(n)$-localization, or climb their way up the ladder of the ANSS, computational efforts work fine at first but then promptly grind to a halt at about 4 steps in. As a result, most concrete computations (like the one in my paper) take place on a lower floor.

Key Idea 2.1. *The ANSS for the sphere and the Morava K-theories $K(n)$ (and therefore the organizational structure of our apartment analogy) are underlain by the theory of formal group laws* [**Haz12**]. *The link between formal group laws and stable homotopy originated in the work of Novikov and Quillen* [**Qui69**]. *The increase in computational complexity as n increases can be tied to an increasingly difficult sequence of group cohomology computations, where the groups in question are automorphism groups of certain formal group laws of increasing complexity (i.e., height—see Definition A2.2.7 in* [**Rav86**]*).*

In my paper, I wander the 2nd floor of this apartment building hoping to learn more about its 3-torsion residents. Amazingly, most of the 2nd floor inhabitants are related to one another—that is, they are a *family*—that homotopy theorists call the beta family. There is a conjectured link between the 3-torsion members of this family and modular forms with certain properties. We are hopeful that our computation may help elucidate this link in future work (at primes ≥ 5 this link is known to exist). Even more amazingly, there are analogous families of elements on floor 1 and on the higher floors with either proven or conjectured links to certain number-theoretic objects. These ideas lie on the cutting edge of homotopy theory. We'll take a tour of the entire building (to the extent we can) from the bottom up, with an eye toward this cutting-edge technology. This should help put my paper in its proper context.

2.2. The lobby. The zeroth Morava K-theory $K(0)$ is equivalent to rational homology $H\mathbb{Q}$. This means $L_{K(0)}S$ detects non-torsion, i.e., it detects $\pi_0 S \cong \mathbb{Z}$. Therefore the lobby-level residents are the integers! No wonder they're so friendly and inviting.

2.3. The first floor. In [**Ada66**] Adams used K-theory to compute the image of the stable J-homorphism

$$J : \pi_*(SO) \to \pi_* S$$

from the homotopy of the stable orthogonal group to the stable stems. The elements of the image of J all live on floor 1 and are generated by a family of related elements known as the alpha family, the first and most thoroughly understood of the *Greek letter families*. The $K(1)$-local sphere ($K(1)$ itself being a summand of K-theory completed at p) zooms in on the alpha family elements. We will linger on floor 1 for a bit to describe how the alpha family is organized, since the higher Greek letter families have similar organizational structures. Might as well enjoy some wine and cheese, too.

It is easiest to first fix a prime p and discuss the p-torsion of the alpha family, so let's do it. Then in the ANSS there is a collection of elements $\{\alpha_i^{alg}\}$ indexed by integers $i \geq 1$, where

$$\alpha_i^{alg} \in \mathrm{Ext}_{BP_*BP}^{1,2pi-2i}(BP_*, BP_*)$$

so these elements live on the 1-line. In the chromatic spectral sequence (CSS), which is an SS converging to the ANSS E_2-page that we will treat like a black box in this discussion, α_i^{alg} is born out of a "fraction" of the form

$$\frac{v_1^i}{p}$$

and so the subscript i determines the power on v_1. Because you can go from one member of the collection to the next by multiplying by v_1 in the CSS, this collection is called a v_1-periodic family (think of Bott periodicity). They are all non-trivial elements of the ANSS E_2-page; in fact, they all have order p.

The ANSS is such an algebraic jungle, that the existence of this infinite yet intimately connected network of nontrivial elements is miraculous. But it gets better. It turns out each α_i^{alg} survives the ANSS and yields a homotopy element $\alpha_i \in \pi_{2pi-2i-1}S$. So the *alpha family at* p, i.e., the collection $\{\alpha_i^{alg}\}$, yields a corresponding collection $\{\alpha_i\} \subset \pi_*S \otimes \mathbb{Z}_{(p)}$. Taking the union of these latter sets over all primes yields the family of wine and cheese enthusiasts that occupies most of floor 1 of the building.

Recall that if

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

then B_n is the n-th *Bernoulli number*. The sequence $\{B_n\}$ is significant in number theory, from summing up the m-th powers of the first k integers to Fermat's Last Theorem, and many things in between. There turns out to be a close relationship between the alpha family at odd primes (the situation at $p = 2$ is muddier) and Bernoulli numbers.

KEY IDEA 2.2. *Let p be any prime.*

(1) *Each member of the alpha family at p yields a non-trivial element of π_*S of order p, and they collectively generate the image of J at p (although if $p = 2$, neither of these assertions is 100% true).*
(2) *The alpha family element α_i^{alg} is divisible by $j - 1$ powers of p, where j is 1 plus the number of powers of p dividing i.*

(3) *If p is odd, there is a correspondence between the alpha family at p and Bernoulli numbers, in the sense that the order p^j of α_i^{alg}/p^{j-1} is the p-factor of the denominator of B_t/t where $t = pi - i$ and B_t is the t-th Bernoulli number.*

2.4. The second floor. For a fixed p, the *beta family at p* $\{\beta_i^{alg}\}$ lives on the 2-line of the ANSS. The provenance of the beta family is analogous to what occurred one floor below with the alpha family, since β_i^{alg} comes from a "fraction"

$$\frac{v_2^i}{v_1 \cdot p}$$

in the CSS. The beta family is therefore a v_2-periodic family.

Ideally, the beta family $\{\beta_i^{alg}\}$ across all primes would consist entirely of non-trivial elements that survive the ANSS and yield non-trivial homotopy elements $\{\beta_i\} \subset \pi_* S$. But these are the same $\beta_i s$ that couldn't wait to boot you out the door! We know they're a tad dodgy, and therefore unlikely to exhibit behavior as consistent as the alpha family. This intuition is sound. In fact, at the very first prime ($p = 2$), the very first beta element

$$\beta_1^{alg} \in \mathrm{Ext}^{2,4}_{BP_*BP}(BP_*, BP_*)$$

is zero! To guarantee all beta elements in the ANSS are non-trivial, p must be at least 3, and to guarantee they all yield non-trivial homotopy elements p must be at least 5.

Recall that a *modular form over $\mathbb{Z}$* is a function $f : \mathfrak{h} \to \mathbb{C}$ on the upper half plane $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ satisfying

$$f(\gamma z) = (cz + d)^k f(z)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, as well as a growth condition at $i\infty$. The "over $\mathbb{Z}$" part means that the Fourier expansion obtained from the nice periodicity property $f(z + 1) = f(z)$, namely

$$f(q) = \sum_{i=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$, has integer coefficients. Modular forms must satisfy so many symmetries simultaneously that their existence is miraculous, but they're out there. And, like Bernoulli numbers, the Fourier coefficients of modular forms encode a lot of number-theoretic information. It turns out that modular forms are to the beta family what Bernoulli numbers are to the alpha family.

Key Idea 2.3. (1) *If $p \geq 5$, each member of the beta family at p yields a non-trivial element in $\pi_* S$ of order p.*

(2) *Certain beta elements β_i^{alg} have representatives in the CSS that can be further divided by v_1 and p, and if $p \geq 5$, there is a 1-1 correspondence*

between these "divided" beta family elements and modular forms over $\mathbb{Z}$ up to certain congruences depending on p.

(3) *In my paper I give evidence that $\pi_* Q(2)$ detects "divided" beta family elements at the prime 3 by finding candidate detecting elements on the level of Adams-Novikov E_2-terms. My hope is that this will eventually lead to a 3-primary version of the aforementioned 1-1 correspondence.*

The correspondence with modular forms is due to Behrens [**Beh09**], and is one of my main motivations for studying $Q(2)$ at the prime 3. Moreover, in the course of proving this 1-1 correspondence Behrens shows that the divided alpha and beta families at $p \geq 5$ are detected by the $E(2)$-localized unit map

$$\pi_* L_{E(2)} S \to \pi_* Q(\ell)$$

for appropriate values of ℓ depending on p. We would like to know whether something analogous is true at $p = 2$ and $p = 3$.

As I mentioned above, modular forms actually exist, and now is a great time to exhibit one. For a positive integer t consider the q-expansion

$$E_t(q) = 1 - \frac{2t}{B_t} \sum_{n=1}^{\infty} \sigma_{t-1}(n) q^n$$

where $\sigma_{t-1}(n)$ is the sum of the $(t-1)$-st powers of the divisors of n. If $t \geq 4$ then $E_t(q)$ is in fact a modular form of weight t called the *Eisenstein series of weight* t. If $p \geq 5$ then the Eisenstein series E_{p-1} (called the *Hasse invariant*) has the property that $E_{p-1} \cong 1 \mod p$, which means multiplication by E_{p-1} (modular forms have a ring structure—they can be added and multiplied) takes you from weight k modular forms to weight $k + p - 1$ modular forms without changing the q-expansion modulo p. This of course produces *congruences* between the q-expansions of modular forms, and turns out to be a key ingredient of Behrens' proof in [**Beh09**]. Unfortunately the q-expansion E_{p-1} doesn't quite accomplish this if p is 2 or 3. For example, while E_2 can be regarded as a modular form in a certain sense (different from the sense discussed here) it does not have the requisite properties to help produce congruences. The hope is that $Q(2)$ can come to the rescue, at least the prime 3.

2.5. The third floor and higher. The pattern of the Greek letter family construction might now be clear. The gamma family $\{\gamma_i^{alg}\}$ at p lives on the 3-line of the ANSS and is a v_3-periodic family, as each γ_i^{alg} originates as a fractional element

$$\frac{v_3^i}{v_2 \cdot v_1 \cdot p}$$

in the CSS. The somewhat-unpredictable behavior exhibited by the beta family continues here; for example, at the prime 2 the element

$$\gamma_1^{alg} \in \mathrm{Ext}^{3,*}_{BP_*BP}(BP_*, BP_*)$$

is zero (more on γ_1^{alg} very soon—stay tuned), and γ_3^{alg} does not survive the ANSS at the prime 5 because it is the source of a nontrivial differential on the 33rd page. However, things once again stabilize if the prime is large enough.

Key Idea 2.4.
(1) *If $p \geq 7$, each member of the gamma family at p yields a non-trivial element in $\pi_* S$ of order p.*
(2) *The behavior of the Greek letter families we've discussed becomes more regular as the prime increases because of sparseness. The larger the prime, the more spread out the elements of the ANSS are, which in turn means less action with the differentials and more predictable results.*

The element γ_1 (say, at any $p \geq 7$ to be safe) may go down in mathematical history as the biggest troublemaker in the entire apartment buliding. Why? Because γ_1 caused (intentionally, no doubt) significant confusion in the homotopy theory community in the early 1970s. At that time, Shichiro Oka and Hiroshi Toda announced that $\gamma_1 \in \pi_* S$ is zero at a conference in Japan. Also at that time, Emery Thomas and Raphael Zahler announced that $\gamma_1 \neq 0$ in a paper in the Journal of Pure and Applied Algebra [**TZ74**]. Outstanding mathematicians, the four of them, and γ_1 remained elusive, like a celebrity outsmarting the paparazzi. And if that wasn't bad enough, journalists from *Science* [**Sci**] and *The New York Times* [**NYT**] took this snafu in homotopy theory as an opportunity to declare that the decline of mathematics was inevitable! (Slow news cycle?)

Eventually, with the help of Frank Adams, the situation was sorted out and γ_1 is indeed nontrivial in homotopy for $p \geq 7$. But there are still way more questions than there are answers on floor 3. For example, what number-theoretic objects (if any) do the gamma family elements naturally pair up with? And then there are the higher floors, too. What is true about the deltas on floor 4? The epsilons on floor 5? The zetas on floor 6? I don't believe I've ever seen the latter two families even mentioned in print, though I'm sure my colleagues will correct me if I'm wrong. Oh, and what are their number-theoretic counterparts, by the way? And what happens when Greek alphabet is exhausted? Well, almost nothing is known about these higher families or about the $K(n)$-local sphere for $n \geq 4$, whether at small primes or large ones. Nonetheless, homotopy theorists work around the finiteness of the Greek alphabet by letting $\alpha^{(n)}$ denote the n-th Greek letter, so we're covered either way.

A glimmer of hope appears in a recent manuscript by Behrens and Lawson [**BL10**] in which they construct *topological automorphic forms* (or TAF for short), a higher analog of topological modular forms. According to their theory, gaining access to the higher floors of the building requires replacing elliptic curves with higher-genus objects called *Shimura varieties*; it requires replacing modular forms with more general objects called *automorphic forms*; and analogs of $Q(2)$ are conjectured to exist as well. The amount of information required to just *define* TAF is daunting, so while progress in this direction is possible and exciting to think about, it is likely to be slow.

What do you suppose is happening, geometrically or algebraically or in any other respect, with the omicron family (floor 15) at the prime 691? Or, how about the 25th floor, where the Greek letters run out? Or the seven millionth floor? The mind reels.

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