Enchiridion: Mathematical User's Guides
`mathusersguides.com`
Volume 3 (September 2017)

A user's guide: The Adams-Novikov E_2-term for Behrens' spectrum $Q(2)$ at the prime 3

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3. Story of the development

To help myself write about the development of my paper, I dusted off my research notebook from graduate school. It had been a few years since I last looked at it (shame on me). Reading it now brings back a flood of memories. The first few pages call to mind both the excitement I felt starting on my thesis project, and the frustration I felt while being stuck in the mud on numerous occasions. The last few pages, in which my computation finally began to come together, chronicle the sense of relief I felt from knowing I was near the finish line. Something else I noticed—immaterial to anyone besides me, but I'll mention it anyway—is that I used to be a much better note-taker. I used complete sentences, had neat handwriting, and employed different colors of ink for contrast. These days, I write my notes in chicken scratches with a dull pencil that's been thoroughly chewed by my kids. When I'm done writing this I'm buying some new pens. But I digress.

My old research notes reveal that the development of my paper occurred roughly in three stages. In stage 1, I studied the number-theoretic properties of Q-spectra at primes greater than 3 and tried to prove their 3-primary analogs, but failed. In stage 2, still motivated by number-theoretic considerations but unable to make progress on them directly, I turned my attention to simpler, purely computational questions about the homotopy of $Q(2)$ at the prime 3. Finally, stage 3 began with a key breakthrough that I used to carry out my ANSS computation.

3.1. Stage 1. At the start of Topic 1, I noted that the chromatic convergence theorem of Hopkins and Ravenel implies π_*L_2S is a second-order approximation to $\pi_*S\otimes\mathbb{Z}_3$. The theorem itself is actually broader. It says that if X is a spectrum and L_n is localization at the n-th Johnson-Wilson theory $E(n)$ at the prime p, then there is a tower (called the *chromatic tower*)

$$L_0X \leftarrow L_1X \leftarrow L_2X \leftarrow L_3X \leftarrow \cdots$$

whose homotopy limit is (under mild hypotheses on X) the p-localization of X. In particular, the theorem holds for $X = S$ and $p = 3$, yielding the assertion.

Given a map $L_nX \to L_{n-1}X$ from the chromatic tower, its fiber M_nX is the n-th *monochromatic layer* of X. These monochromatic layers sit atop the chromatic tower, as follows:

$$\begin{array}{ccccccccc}
 & & M_1X & & M_2X & & M_3X & & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
M_0X = L_0X & \longleftarrow & L_1X & \longleftarrow & L_2X & \longleftarrow & L_3X & \longleftarrow & \cdots
\end{array}$$

If $X = S$ then π_*M_nS zooms in on the n-th floor residents of the metaphorical homotopy apartment building from Topic 2. Moreover, there is a *chromatic spectral sequence* that stitches all of this data together, and it has the form

$$E_1^{s,t} = \pi_t M_s X \Rightarrow \pi_{t-s} X_{(p)}.$$

Behrens proves his correspondence between modular forms and beta elements at primes $p \geq 5$ (as well as other results) by studying $Q(\ell)$ from two perspectives. On the one hand, he studies the chromatic tower of $Q(\ell)$, which ends at $n = 2$ since $Q(\ell)$ is $E(2)$-local. On the other hand, he studies the semi-cosimplicial group $C(\ell)^\bullet$ obtained by applying the functor π_* to the semi-cosimplicial diagram of spectra underlying $Q(\ell)$. [The bullet in the notation $C(\ell)^\bullet$ is a placeholder for dimenison, so $C(\ell)^0 = \pi_*TMF$, $C(\ell)^1 = \pi_*(TMF_0(2) \vee TMF)$, and $C(\ell)^2 = \pi_*(TMF_0(2))$.] These perspectives turn out to be closely linked. We will now sketch the logical chain of ideas that mediates between the two perspectives and yields Behrens' correspondence.

Key Idea 3.1. *Fix a prime $p \geq 5$ and let i range from 0 to 2.*

1. *Since the spectra $Q(\ell)^i$ are built from variants of TMF, the groups $C(\ell)^i$ comprise modular forms.*
2. *For each i there is a spectral sequence whose input is the cohomology of an arithmetically altered version of $C(\ell)^\bullet$, and whose output is $\pi_*M_iQ(\ell)$.*
3. *By sparseness in the aforementioned spectral sequences, there are isomorphisms linking the cohomological data (i.e., modular forms) to the homotopy of the chromatic layers (see* [**Beh09**]*, Corollary 7.7).*
4. *The maps on modular forms given by multiplication by powers of the Eisenstein series E_{p-1} are inclusions, which can be transported over to the monochromatic layers thanks to the aforementioned isomorphisms, where Behrens can leverage them to prove his correspondence (see* [**Beh09**]*, Theorem 11.3).*

Part of what goes wrong with the modular form $E_{3-1} = E_2$ at the prime 3 is that it is not symmetric with respect to the action of the full modular group $\mathrm{SL}(2, \mathbb{Z})$; rather, it is symmetric only with respect to a certain subgroup of $\mathrm{SL}(2,\mathbb{Z})$ (something I eluded to in the previous section). There may be a way around this difficulty with a clever modular form argument. I tried for many months to find such an argument, but at the time I was (and in fact, still am) unable to do so! Something that would have been natural for me to explore during this period is the possibility of explicitly computing $\pi_* M_i Q(2)$ at $p = 3$ for $i = 0, 1, 2$. Maybe I will put my new pens to good use and try it.

3.2. Stage 2. Mid-way through my thesis research, when I was stuck in the manner I described above, I went to visit Mark Behrens in Boston for two weeks. The visit with Mark helped. Perhaps that is not surprising. But, the *way* in which it helped was unexpected—it shifted my focus toward more fertile ground.

The first thing Mark and I discussed was how one might obtain the 3-primary analog of his correspondence, since that has been at the forefront of my mind. Around that time, Mark had consulted a few modular forms experts and they were not sure how to tweak the argument. Mark himself was also not totally sure, although he had some rough ideas and we talked about them. The notes I have from those discussions are technical and I am still parsing through them to this day. However, as our conversations continued, we focused more and more on pure spectral sequence computations for spectra such as TMF and $Q(2)$, as well as the monochromatic layers $M_i Q(2)$.

The ease and fluidity with which Mark drew and dissected spectral sequence charts made it all seem so natural. I had worked through spectral sequence computations before (awkwardly and with a lot of struggle, like most beginners) but Mark got me hooked on them. My desire to get better at basic computations grew, and I began to put considerations about modular forms off to the side. I thought (rightly so!) that I could always return to the number theory questions down the road. I eventually settled on attacking the ANSS for $Q(2)$ and seeing how far I could get with it.

Behrens' explicit description of the cosimplicial structure underlying $Q(2)$ in [**Beh06**] was all anyone needed to embark on this computation. So, when I returned from Boston, I got to work. According to my old research notebook I initially made a lot of progress. But soon after, the computation became unwieldy in a way that I will describe in the next subsection. I was stuck once again.

3.3. Stage 3. The breakthrough that got me unstuck came thanks to a short note Behrens wrote—for his own use, not necessarily for publication—on how to compute the rational homotopy of $Q(2)$ [**Beh**]. He had shown me this note briefly during my visit but I did not read it carefully at that time. In it, Behrens computes the cohomology of the same three-term complex we discussed in Section 1, but *rationally*, on his way to obtaining the result. Behrens ultimately shows that the rational homotopy of $Q(2)$ is concentrated in degrees -2, -1, and 0, meaning $\pi_* Q(2)$ is mostly torsion, as is the case for $\pi_* L_{K(2)} S$. This is evidence that $Q(2)$ does a great job capturing what is happening with the $K(2)$-local sphere.

The methods Behrens uses are from linear algebra, since all the modules in sight are over a field ($\mathbb{C}$, as happens to be the case in Behrens' exposition). Even though working rationally makes computing kernels and cokernels easier, Behrens has to do a decent amount of work to nail things down. The keys to his success are cleverly filtering the diagram, and making judicious choices of $\mathbb{C}$-bases for B, MF, and their various quotients.

My key insight was to observe that Behrens' filtration and his choices of basis could still be used profitably 3-locally.

Key Idea 3.2. *In the diagram of Hopf algebroids*

$$(B, \Gamma) \rightrightarrows (B, B) \oplus (B, \Gamma) \Rrightarrow (B, B)$$

the coface maps are maps of $\mathbb{Z}_{(3)}$*-algebras, and are determined by their action on polynomial generators. But the alternating sums of coface maps are* $\mathbb{Z}_{(3)}$*-module maps only, so one has to compute what they do to monomials, and the formulas become complex. Those formulas remain complex after passing to the cochain complex*

$$MF \to B \oplus MF \to B$$

in the DCSS (see Topic 1). Behrens encounters a similar difficulty in his rational computation, and proceeds by first filtering the complex, and then choosing $\mathbb{C}$*-bases for the rationalized versions of* B *and* MF *that drastically simplify computations within the associated graded. Those choices can be mimicked in the 3-local setting.*

In other words, much of the computational work in my paper could be characterized as "pseudo-linear algebra," which in this case means linear algebra without inverting 3.

Let us look at an example. Behrens' filtration applied to the cochain complex from my computation takes the form

$$(MF \to B \oplus MF \to B) \supset (MF \to MF \to 0) \supset (0 \to 0 \to 0).$$

After passing to the associated graded, the data necessary to move forward includes the kernel and cokernel of a map $\psi_d + 1 : B \to B$ where 1 is the identity on B and ψ_d is an algebra homomorphism defined by

$$\begin{aligned} \psi_d : q_2 &\mapsto -2q_2, \\ q_4 &\mapsto q_2^2 - 4q_4. \end{aligned}$$

(On the level of Weierstrass equations, ψ_d records the effect of replacing an elliptic curve C by its quotient C/H where H is an order 2 subgroup; or, equivalently, replacing the degree 2 isogeny $C \to C/H$ by its dual isogeny $C/H \to C$, hence the "d" in the notation.)

In Behrens' rational computation, he notes that

$$B \otimes \mathbb{C} = \mathbb{C}[q_2, q_4, q_4^{-1}, \mu^{-1}]$$

and that the set $\{q_4^i \mu^j q_2^\epsilon : i, j \in \mathbb{Z}, \epsilon = 0, 1\}$ is a $\mathbb{C}$-basis for $B \otimes C$. He then breaks up $B \otimes C$ as a direct sum of 2-dimensional subspaces

$$V_{i,j,\epsilon} = \mathbb{C}\{q_4^i \mu^j q_2^\epsilon, q_4^j \mu^i q_2^\epsilon\}$$

each of which is *invariant* under $\psi_d + 1$. Restricting $\psi_d + 1$ to each $V_{i,j,\epsilon}$ yields a 2×2 matrix. Simple.

In my computation, I follow Behrens' lead by noting that $\{q_4^i \mu^j q_2^\epsilon\}$ (where i, j, and ϵ are as above) is a set of generators for B as a $\mathbb{Z}_{(3)}$-module. For computational convenience, I deviate a bit and form the 2-dimensional direct summands

$$V'_{i,j,\epsilon} = \mathbb{Z}_{(3)}\{s^i t^j q_2^\epsilon, s^j t^i q_2^\epsilon\}$$

where $s = 8q_4$ and $t = \mu/8$. They are invariant subspaces of $\psi_d + 1$ and they allow me to study $\psi_d + 1$ as a sequence of 2×2 matrices. In cases where the 2×2 matrix is clearly nonsingular in the rational computation, it is not always so clear 3-locally. I manage to obtain a basis of eigenvectors and compute the eigenvalues, but all the while I must take care to observe whether the eigenvalues are invertible in $\mathbb{Z}_{(3)}$. In fact, they are often not invertible, in which case determining their 3-divisibility is the name of the game.

References

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