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A user's guide: Categorical models for equivariant classifying spaces

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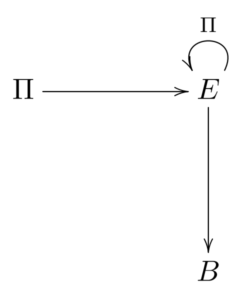
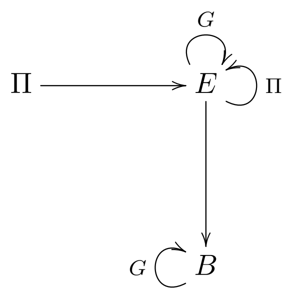
2. Metaphors and imagery

We will start with a remark on notation which was also made in the previous section, but we would like to emphasize it to avoid confusion. Usually when one talks about principal G-bundles, one refers to bundles with structure group G - this means that G acts on the right on the total space such that the action preserves the fibers and is free and transitive on the fibers, which are homeomorphic to G. We will instead use the notation Π for the structure group – this is simply because we will use G for the equivariance group. We can picture a principal Π-bundle as in the following diagram; Π acts on the total space and the fibers are Π. This is so far a nonequivariant principal Π-bundle:

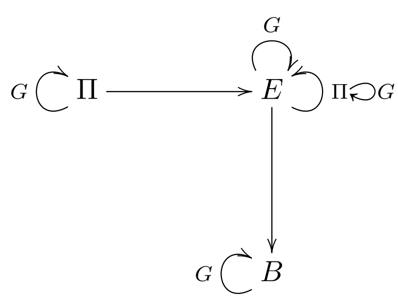


We will introduce the actions of G in two steps, starting with the easier case and then moving on to the more complicated and more general case, so that the reader can slowly build a mental image of where and how G acts in the definition of a G-equivariant principal Π-bundle. Such a bundle will of course, in particular, be a principal Π-bundle.

First, let us think of the "easier" case in which the group G does not act on the structure group Π. In this case a principal G-equivariant Π-bundle, or a principal (G, Π)-bundle, is a principal Π-bundle in which we have a left G-action on the total space which commutes with the Π-action and a G action on the base space such that the projection map is equivariant. With a little bit of algebraic manipulation, it is not hard to see that two commuting actions give rise to an action of the direct product of the two groups, thus on the total space we have an action of $\Pi \times G$. We can picture this by adding G-actions in our previous diagram in the appropriate places -the G-actions do not interact with the Π-action:



The more general and harder case is the one in which the group G acts on the structure group Π. In this case a principal G-equivariant Π-bundle, or a principal (G, Π_G)-bundle, is a principal Π-bundle in which the actions of G and Π on the total space satisfy a *twisted* commutation relation, which translates into an action of the semidirect product (instead of the product) $\Pi \rtimes G$, which again is free when restricted to Π, and the projection map is again G-equivariant. The picture from before just gets a little more complicated as we add a G-action on the fiber Π and twist the actions of the two groups on the total space:



So far we have tried to build the image of a principal (G, Π_G)-bundle when G acts on the structure group Π. Now, nonequivariantly we know that every principal Π-bundle arises as the pullback of the *universal* principal Π-bundle $E\Pi \to B\Pi$ along some classifying map of the base into $B\Pi$. Namely, if $E \to B$ is a principal Π-bundle, E is the pullback of a diagram

$$\begin{array}{ccc} E & \longrightarrow & E\Pi \\ \downarrow & & \downarrow \\ B & \longrightarrow & B\Pi \end{array}$$

This is also true equivariantly: there exists a universal principal (G, Π_G)-bundle $E\Pi_G \to B\Pi_G := E\Pi_G/\Pi$ so that every principal (G, Π_G)-bundle is the pullback of this bundle along an equivariant classifying map. If $E \to B$ is now a principal (G, Π_G)-bundle, it fits in a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & E\Pi_G \\ \downarrow & & \downarrow \\ B & \longrightarrow & B\Pi_G \end{array}$$

Therefore, in order to understand principal (G, Π_G)-bundles, one needs to understand the principal universal bundle $E\Pi_G \to B\Pi_G$.

The main result of this paper gives a model of the universal principal (G, Π_G) bundle in terms of classifying spaces for categories. We try to give some imagery for the categories that show up.

2.1. G-categories. A G-category $\mathscr{C}$ can be thought of as a functor

$$G \to \mathscr{C}at$$

from G, regarded as a one object category to the category of categories. Explicitly, the one object of G maps to $\mathscr{C}$, and each morphism of G, namely each element g, maps to an endofunctor of the category $\mathscr{C}$. Functoriality encodes the usual conditions

$$g(hx) = ghx \qquad \text{and} \qquad ex = x$$

for an object or morphism x. But also, composition needs to be respected:

$$g(f \circ f') = gf \circ gf'$$

for morphisms f, f' in $\mathscr{C}$ since each g is a functor $\mathscr{C} \to \mathscr{C}$.

Thus we can visualize a G-action on a category as permuting both the objects and the morphisms. For any subgroup $H \subseteq G$, the fixed points $\mathscr{C}^H$ are the subcategory formed by those objects and morphisms that are fixed under the action of all the elements of H.

2.2. The category $\widetilde{G}$. One of they key players is the category $\widetilde{G}$, which is defined for any group G to have as objects the elements of G and exactly one morphism between any two objects. This means in particular that any two objects are isomorphic to each other via a unique isomorphism. This is an instance of the more general concept of *chaotic or indiscrete category* defined for any space to have as objects the points in the space and a unique morphism between any two objects.

The category $\widetilde{G}$ is G-isomorphic to the translation category of G –they both have as objects the elements of G on which G acts by translation and since there is a unique morphism between any two objects in both of these categories, the action on objects completely determines the action on morphisms. The only reason why we are saying that $\widetilde{G}$ and the translation category of G are only G-isomorphic and not equal is that the actions are not identical as a result of the choice of labeling of the morphisms. In $\widetilde{G}$, the morphisms are labelled by source and target, namely the unique morphism from g to h is labelled by (h, g), which forces the action to be diagonal on morphisms. (The reversal of source and target is only to make composition more transparent.) In the translation category of G, on the other hand, the morphism labelled (g, h) stands for the morphism starting at h and ending at gh, which forces the action on morphisms to be on the right.

A key feature of the category $\widetilde{G}$ is that it is contractible: since there is a unique morphism between any two objects, any object is initial and terminal. Therefore the classifying space of this category $B\widetilde{G}$ is contractible. Also, since the action on $\widetilde{G}$ is free, the action on the classifying space is also free. So $B\widetilde{G}$ is a contractible space with a free G-action, which means that it is equivalent to EG.

Note that $\widetilde{G}$ is *not* equivariantly contractible – that would mean that all the fixed point categories are contractible, but $\mathscr{C}^H$ for $H \neq G$ is empty, so it is not contractible.

2.3. The category $\mathscr{C}at(\widetilde{G}, \mathscr{C})$. Suppose $\mathscr{C}$ is a G-category. Then $\mathscr{C}at(\widetilde{G}, \mathscr{C})$ is the category of *all* functors $\widetilde{G} \to \mathscr{C}$ and natural transformations between these. We emphasized the word "all" because it is crucial that we are taking all functors, and natural transformations and not just the equivariant ones in order to define a G-action on the category $\mathscr{C}at(\widetilde{G}, \mathscr{C})$. The G-action on $\mathscr{C}at(\widetilde{G}, \mathscr{C})$ is by conjugation, so the fixed points $\mathscr{C}at(\widetilde{G}, \mathscr{C})^G$ of this G-category are precisely the equivariant functors and natural transformations because those are the ones that are invariant under conjugation.

The objects of $\mathscr{C}at(\widetilde{G}, \mathscr{C})$ are not hard to visualize: a functor $\widetilde{G} \to \mathscr{C}$ is just a diagram of objects in $\mathscr{C}$ indexed over the elements of G together with a unique isomorphism between any two of them.

2.4. The model for equivariant bundles and a reality check. The model that we find in the paper for the universal principal (G, Π_G)-bundle when G is finite or discrete, but Π is allowed to be compact Lie, is

$$B\mathscr{C}at(\widetilde{G}, \tilde{\Pi}) \to B\mathscr{C}at(\widetilde{G}, \Pi).$$

Maybe this is not that easy to visualize, but let us just do a reality check: If we forget the G-actions altogether, this is supposed to be a universal Π-bundle.

Since $\widetilde{G}$ is a contractible category nonequivariantly , we have nonequivariant equivalences of categories

$$\mathscr{C}at(\widetilde{G}, \tilde{\Pi}) \simeq \tilde{\Pi} \quad \text{and} \quad \mathscr{C}at(\widetilde{G}, \Pi) \simeq \Pi,$$

and since $B\tilde{\Pi} \simeq E\Pi$, nonequivariantly this bundles is equivalent to

$$E\Pi \to B\Pi.$$

The last image we want the reader to keep in mind is that a universal principal (G, Π_G)-bundle is nonequivariantly the universal Π bundle $E\Pi \to B\Pi$, but with some funky G-actions everywhere. Replacing the categories $\tilde{\Pi}$ and Π by the nonequivariantly equivalent categories $\mathscr{C}at(\widetilde{G}, \tilde{\Pi})$ and $\mathscr{C}at(\widetilde{G}, \Pi)$ builds in "the right equivariant homotopy type." We haven't discussed in this users guide what this "right" equivariant homotopy type is, but let us just say here that there is an explicit characterization of the $\Pi \rtimes G$ homotopy type of the total space in a universal (G, Π_G)-bundle that one can check, just like there is a characterization of the Π-equivariant homotopy type of $E\Pi$: the fixed point space for the trivial subgroup $E\Pi^e = E\Pi$ is contractible and all the other fixed point subspaces $E\Pi^H$ for subgroups $H \neq e$ are empty.

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