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A user's guide: Monoidal Bousfield localizations and algebras over operads

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1. Key insights and central organizing principles

A key task of mathematics is to provide unifying perspectives. Such perspectives help us to build bridges between seemingly disparate fields, to work in a general setting so that our results hold in numerous specific instances, and to understand the humanistic and aesthetic side of our research. This allows us to determine what drives us to research the topics we choose, which deep results we are truly pursuing, and which ideas the human mind keeps returning to in different guises and different times. One such fundamental idea is:

ORGANIZING PRINCIPLE 1.1. *Whenever possible, one should work in a setting where it is possible to replace the objects of interest by nicer objects which are equivalent in a suitable sense.*

Another fundamental idea is that of localization, which allows us as mathematicians to zoom in on the pertinent information within a problem. This is like putting on glasses to change which objects we view as equivalent, and is familiar to topologists who freely pass between the ordinary category of topological spaces and its homotopy category, where two objects are equivalent if one can be continuously deformed into the other.

My paper [**Whi14b**] is fundamentally about localization. The purpose is to understand how the type of localization which arises in homotopy theory (specifically, Bousfield localization of model categories) interacts with the monoidal structure which must be present in order to "do algebra." The **main goal** of the paper is to find checkable conditions so that Bousfield localization preserves algebraic structure on objects. Doing this in a unified framework allows me to prove new results about equivariant spectra, ideals of ring spectra, and different models for monoidal stable homotopy theory, and to recover several classical results about spaces and chain complexes (see Theorem 1.9 and the discussion afterwards below). Formally, we use operads to encode algebraic structure, and our main theorem gives conditions on a model category $\mathcal{M}$, on an operad P, and on a localization L so that if E is a P-algebra then its localization $L(E)$ is a $\mathcal{P}$-algebra, at least up to weak equivalence. Operads are vehicles for encoding algebraic structure in general monoidal categories, such as commutative structure, associative structure, or Lie algebra structure. En route to proving the main result we provide conditions so that the model category $L(\mathcal{M})$ obtained via Bousfield localization satisfies various axioms that are common in monoidal model category theory. This includes conditions so that commutative monoids inherit a model structure and are preserved under localization. I then check these conditions in numerous model categories of interest, and obtain specific results in all those settings.

In this paper localization arises in two different but related ways. In the first, localization is applied to a model category in order to obtain its homotopy category (a good example to keep in mind is the mental shift you do when thinking of topological spaces up to homeomorphism vs. up to homotopy equivalence, i.e. up to continuous deformation). This form of localization goes back to [**GZ67**] at least, and can be viewed as a generalization of the localization which arises in algebra. There, one localizes by formally adjoining multiplicative inverses to a specified set of elements in a ring. In the category theoretic version one does not have elements to invert, so instead one formally adjoins morphisms which are inverses to a specified class of maps. To recover the ring-theoretic notion of localization one inverts endomorphisms of the ring which correspond to multiplying by the elements one seeks to invert.

Unfortunately, not all choices of sets of maps admit localization. The notion of a model category (which comes equipped with a chosen class of maps to invert called the weak equivalences) arose to fix this issue and to provide control over the morphisms in the localized category. Having the structure of a model category allows for the tools of homotopy theory to be applied, and in this way parts of homological algebra, algebraic geometry, representation theory, logic, graph theory, and even computer science can be viewed as special cases of homotopy theory. To summarize:

Organizing Principle 1.2. *In settings where one has a notion of weak equivalence or something like a homology theory to compress complicated information into simple information, one should try to build a model structure so that the tools of abstract homotopy theory can be applied.*

The other type of localization in this paper is called Bousfield localization, and it is a procedure one applies to a model category $\mathcal{M}$ in order to enlarge the specified class of weak equivalences to contain some specified set of maps C (the resulting model category is denoted $L_C(\mathcal{M})$), see [**Bou75**], [**Bou79**], and [**Hir03**]. Both forms of localization can be viewed as special cases of Organizing Principle 1.1, because both satisfy a universal property saying they are the "closest" to the given category in which the prescribed maps have been inverted. Formally, this means any functor out of the given category which inverts the maps factors through the localization. The notions of monoidal model categories, operads, and Bousfield localization are recalled in Section 2 of [**Whi14b**].

This paper began out of a desire to understand an example of a localization which destroys equivariant commutativity (Example 5.7). This example arose during the recent proof of Hill, Hopkins, Ravenel [**HHR14**] of the Kervaire Invariant One Theorem [**HHR11**]. In this paper, the authors needed to know that a particular Bousfield localization of equivariant spectra preserved commutative structure. My paper recovers and generalizes the theorem of Hill and Hopkins [**HH13**] which provided conditions for such preservation to occur. Following Organizing Principle 1.2, my method of proving a preservation result is to try to put model structures on the category of objects with algebraic structure (e.g. commutative ring spectra). The following theorem reduces the question of preservation to a simpler question.

THEOREM 1.3. *Let $\mathcal{M}$ be a monoidal model category, C a set of maps in $\mathcal{M}$, and P an operad valued in $\mathcal{M}$. If P-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit (semi-)model structures such that the forgetful functors back to $\mathcal{M}$ and $L_C(\mathcal{M})$ are right Quillen functors, then L_C preserves P-algebras up to weak equivalence. For well-behaved P there is a list of easy to check conditions on $\mathcal{M}$ and C guaranteeing these hypotheses hold.*

This theorem is proven in Section 3, and while the paper centers around this result (especially, checking its hypotheses) this is not where the hard work is being done. The proof really just involves a diagram chase and cofibrant/fibrant replacements, following Organizing Principle 1.1. The property of being cofibrant should be viewed as being analogous to being a CW complex or a projective module, and these replacements are just like the CW approximation theorem or the existence of projective resolutions. Cofibrant replacement provides a concrete way to realize that every object is weakly equivalent to a cofibrant object (following Organizing Principle 1.1).

Because the proof only relies on (co)fibrant replacements, it works in the context of semi-model categories. Every model category is a semi-model category, but the converse is not true. Semi-model categories satisfy most, but not all, of the model category axioms, but they retain enough structure to do effectively everything one is used to doing in the setting of model categories. Formally, the difference between semi-model categories and model categories is that in a semi-model category:

- One only knows that a trivial cofibration lifts against a fibration if the domain of the trivial cofibration is cofibrant
- One only only has factorization of a map f into a trivial cofibration followed by a fibration if f has cofibrant domain.

In order to obtain concrete, recognizable results, the paper specializes to two settings: where P is a Σ-cofibrant operad and when P is the Com operad. In the former case it has long been known how to transfer semi-model structures from $\mathcal{M}$ to P-algebras, and Theorem 5.1 recalls the procedure. In the latter case, my companion paper [**Whi14a**] solved the problem and the main result of that paper is recalled in Theorem 6.2. In both cases the key point is that

Key Idea 1.4. *In order to transfer a model structure to a category of algebras over a monad one must have good homotopical control over the free algebra functor. This often requires some kind of filtration so that free extensions (i.e. pushouts) in the category of algebras can be computed by some transfinite process in the underlying category $\mathcal{M}$.*

These filtrations often take many pages to develop, but they are not the main point of any such paper. They are more an artifact of the topologist's method of proof, which involves building a complicated machine to compute something via a transfinite process and then recovering a result of interest as a special case. The Σ-cofibrancy hypothesis effectively ensures good homotopical control in any cofibrantly generated model category $\mathcal{M}$. The Com operad is not Σ-cofibrant, but one can still obtain good control by making a hypothesis on $\mathcal{M}$. In [**Whi14a**] the hypothesis is introduced as the Commutative Monoid Axiom, and has to do with the free commutative monoid functor $Sym(X) = S \coprod X \coprod X^{\otimes 2}/\Sigma_2 \coprod X^{\otimes 3}/\Sigma_3 \coprod \dots$ where S is the monoidal unit and Σ_n is the symmetric group on n letters. The following is proven in [**Whi14a**] and recalled as Theorem 6.2 in [**Whi14b**]:

Theorem 1.5. *If a monoidal model category $\mathcal{M}$ satisfies the commutative monoid axiom (i.e. for any trivial cofibration g, the map $g^{\square n}/\Sigma_n$ is a trivial cofibration) then commutative monoids inherit a semi-model structure from $\mathcal{M}$ which is a model structure if $\mathcal{M}$ satisfies the monoid axiom from* [**SS00**].

The commutative monoid axiom is verified in [**Whi14a**] for model categories of spaces, simplicial sets, chain complexes in characteristic zero, diagram categories, ideals of ring spectra, and positive variants of symmetric, orthogonal, equivariant, and motivic spectra. Theorems 5.1 and 6.2 cover the hypotheses of Theorem 1.3 about P-alg($\mathcal{M}$) having a semi-model structure. In order to check the hypotheses about P-alg($L_C(\mathcal{M})$) we need in addition

Key Idea 1.6. *In order for Bousfield localization to preserve operad algebra structure one should verify that it respects the axioms of monoidal model categories, e.g. the Pushout Product Axiom, the Unit Axiom, the axiom that cofibrant objects are flat, the Commutative Monoid Axiom, and the Monoid Axiom.*

If $L_C(\mathcal{M})$ satisfies all these axioms then Theorem 1.3 will prove that commutative monoids and algebras over Σ-cofibrant operads are preserved by the localization. Section 4 introduces a hypothesis on the maps to be inverted which guarantees the first three of these axioms hold for $L_C(\mathcal{M})$. Indeed, Section 4 characterizes the localizations for which these axioms hold:

THEOREM 1.7. *Assume $\mathcal{M}$ satisfies the pushout product axiom and that cofibrant objects are flat. Then $L_C(\mathcal{M})$ satisfies these axioms (hence the unit axiom too) if and only if for all cofibrant K, all maps of the form $f \otimes id_K$ for $f \in C$ are weak equivalences in $L_C(\mathcal{M})$. If $\mathcal{M}$ has generating cofibrations I with cofibrant domains then it is sufficient to check the condition for K in the set of (co)domains of I.*

This theorem requires a fair bit of work to prove, but it is fun work for model category theorists. The case for cofibrant domains required the nifty Lemma 4.13, which I hope will help future users of model categories. With Theorem 1.7, Theorem 5.1, and Theorem 1.3 we have a list of checkable conditions so that localization preserves algebras over Σ-cofibrant operads. The conditions are checked for numerous examples in Section 5 and recover examples of Farjoun and Quillen for spaces and chain complexes respectively. Counterexamples are also given, including perhaps the first explicit example where the pushout product axiom fails to be satisfied for some $L_C(\mathcal{M})$. Most of the work in the paper comes in checking the examples in Sections 5 and 7, since this is the only way I could tell if the hypotheses I introduced were good or not. This theorem demonstrates another key idea which is in the background of this work

KEY IDEA 1.8. *The theory of monoidal categories can serve as a useful guide when proving results about monoidal model categories.*

In particular, a similar characterization to that in Theorem 1.7 appeared in [**Day73**]. The condition precisely ensures that the localization respects the monoidal structure, and such localizations are dubbed Monoidal Bousfield Localizations in Section 4. Similarly, to check the commutative monoid axiom one must know that localization respects the functor Sym. This helps us check another condition in Key Idea 1.6, and appears in Section 6:

THEOREM 1.9. *Assume $\mathcal{M}$ is a well-behaved monoidal model category satisfying the commutative monoid axiom. Suppose that $L_C(\mathcal{M})$ is a monoidal Bousfield localization. Then $L_C(\mathcal{M})$ satisfies the commutative monoid axiom if and only if* $\mathrm{Sym}(f)$ *is a C-local equivalence for all $f \in C$.*

This result required a great deal of work to prove and was not satisfying to me because of the hypotheses required (currently hidden under the phrase "well-behaved"). I hope to return to this result in the future and get a slicker proof without these hypotheses. With Theorems 1.3, 1.5, and 1.9 we have achieved our goal of finding checkable conditions so that localization preserves commutative monoids. The conditions are checked for numerous examples in Section 7, including spaces, chain complexes, various models of spectra, and equivariant spectra. Results of Farjoun, Quillen, and Casacuberta, Gutierrez, Moerdijk, and Vogt are recovered and generalized, as well as new results for equivariant spectra.

Lastly, in order to make a complete story, Section 8 provides conditions on $\mathcal{M}$ and C so that $L_C(\mathcal{M})$ satisfies the monoid axiom. This is not necessary for

Theorem 1.3 because semi-model structures suffice, but I felt any good theory of monoidal Bousfield localization should include results about the monoid axiom in case users of the paper need full model structures rather than semi-model structures. This section introduces a new axiom called h-monoidality, independently discovered in [**BB14**], and checks it for a wide variety of model categories.

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