Enchiridion: Mathematical User's Guides Volume 3 (September 2017)
`mathusersguides.com`

A user's guide: A monoidal model for Goodwillie derivatives

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3. Story of the development

Many people say math research involves heaps of mistakes, but we don't often talk about them. I'm not going to sugarcoat how much failure this project involved. At the time of this writing, it hasn't made it through peer-review, so stay tuned. The original ideas of this paper came from my advisor, Randy McCarthy. He got the idea of using a homotopy colimit over the category $\mathbb{I}$ from Bökstedt's definition of THH, and he wondered if it could solve some problems in Goodwillie calculus. I spent the first few years of graduate school asking him questions about spectra, functor calculus, and homotopy colimits. We were mostly concerned with the polynomial approximations P_nF. While the homogeneous approximations D_nF were classified by Goodwillie, the polynomial ones have more obtuse classifications. There are results for special cases of functors, and Arone and Ching show that the Taylor tower can be reconstructed from certain information about the derivatives [**AC16**], but an independent classification was wanting.

Thus, we set out on the project of classifying polynomial functors. Randy was fairly confident that it would work, and he led me unknowingly to reprove many results from his paper with Brenda Johnson [**JM03**] but now in a topological setting. When the project was nearly done in summer 2014, I gave two talks about it in Europe. The first was Goodwillie's 60th birthday conference in Dubrovnik. I spent the week before the conference traveling around, stressing out, and proving that some of my results only held for highly connected spaces on a bus through the Croatian countryside. A little fazed, I spoke at the Young Topologists Meeting where a mistake was found by some students in the audience. I spent the rest of the summer slowly pushing around the bits we were unsure about (had I just phrased them wrong?) until they were clearly in contradiction. It took a couple months to convince Randy we were wrong, and I learned a lot more about why it didn't work in this time. We were feeding k-spheres into the main theorem which only held on k-connected spaces. This was an enormous, embarrassing dead end, but I scraped together the measly results I had for posterity.

I was dismayed at having LIED to the experts in my field, my thesis broken, when Randy suggested I learn about operads and show the derivatives of the identity formed one. At the time, operads were a thing I had seen defined in talks, and I was fairly certain that I would never understand them. I was demoralized to work on yet another difficult problem that was already solved by some geniuses in an (at the time, for me) incomprehensible way.

The original goal of the project was to find an operad structure on the derivatives of the identity functor of spaces by applying definitions and using a homotopy colimit over $\mathbb{I}$. The first order of business was to define the cross effects in the right way to get an associative map. They should fit together in cool nesting cubical ways, but somehow I couldn't pin them down in levels higher than 1. So we found an inductive definition of cross effects which seemed to work. The next step was to figure out if the multilinearizations fit into an operad structure. Sometime in here we also hammered out a proof of the chain rule, essentially a fun induction argument that would hold if we could get the operad structure right. I had to figure out what exactly equivariance was for operads. It turns out that there are two equivariance diagrams for operads, which can be consolidated into one.

$$
\begin{array}{ccc}
\mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_k) & \xrightarrow{\sigma\otimes\tau_1\otimes\cdots\otimes\tau_k} & \mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_k) \\
\Big\downarrow{\scriptstyle id\otimes\sigma^*} & & \\
\mathcal{O}(k)\otimes\mathcal{O}(j_{\sigma^{-1}(1)})\otimes\cdots\otimes\mathcal{O}(j_{\sigma^{-1}(k)}) & & \Big\downarrow{\scriptstyle\gamma} \\
\Big\downarrow{\scriptstyle\gamma} & & \\
\mathcal{O}(j_{\sigma^{-1}(1)}+\cdots+j_{\sigma^{-1}(k)}) & \xrightarrow{\sigma_*(\tau_1,\ldots,\tau_k)} & \mathcal{O}(j_1+\cdots+j_k)
\end{array}
$$

Here $\sigma(\tau_1,\ldots,\tau_k)$ is the composite $(\tau_1\oplus\cdots\oplus\tau_k)\sigma_*(j_1,\ldots,j_k)$, where for $\sigma\in\Sigma_k$, $\sigma(j_1,\ldots,j_k)\in\Sigma_j$ permutes blocks of size j_s according to σ, and for $\tau_s\in\Sigma_{j_s}$ $\tau_1\oplus\ldots\oplus\tau_k$ denotes the image of $(\tau_1,\ldots,\tau_k)$ under the natural inclusion of $\Sigma_{j_1}\times\ldots\times\Sigma_{j_k}$ into Σ_j.

After dragging my feet for a while, I checked that $\partial_* Id$ satisfied the complicated diagram, and my map worked swimmingly. Success! Relief! It was around this time that we realized that the result could be phrased in terms of the derivatives functor ∂_* being monoidal, so the proofs were overhauled to fit this general setup. I can't pinpoint the exact moment of this revelation, but it was sometime between January and July 2015 according to my email inbox, and one lead is an email thread with Cary Malkiewich setting up a meeting to talk about the project. My impression is that a lot of people knew the derivatives should be monoidal; it's essentially spelled out in the introduction of [**AC11**], in a future work section teeming with insight and possible projects. When I went to Stockholm in December 2015 to talk to Greg Arone about the project, he showed me that I had not checked the right diagram, and my map actually failed to satisfy the diagram for nontrivial τ's. He was clearly right, and the problem lay in the way the linearizations fit together, that is, the loop sphere problem discussed in Topic 2.

Greg told me that a normal fix for my mistake would be to use a diagonal map, but he warned me that this would not work in my case, and he offered another possible solution, the sphere operad. He mentioned another small oversight about reduced-ness of the cross effects that left my functors without assembly maps which is kind of crucial, but I decided that this was easily fixed. I left Stockholm determined to amend errors and understand Greg's solutions.

I spent the next month trying to understand what would happen if I used the diagonal map. It worked; I could not see what the problem was. I spent the month after trying to fit a complicated associativity diagram onto one page (it was at least 3-dimensional) and convinced multiple mathematicians in Urbana that everything worked, then defended my thesis; it was time to check with Greg. He responded with a clear counterexample showing the map was not associative and a friendly admonition to try the sphere operad.

The sphere operad really did work! I had already deposited my thesis using the incorrect map, and I spent a month checking this last alteration meticulously. Finally, done! I could leave Urbana with peace of mind that my thesis was correct. I had a new collaboration to work on and decided to worry about the publications from the thesis later. In the back of my mind there was a little voice saying, "Are you sure? Didn't Greg say something else?" That pesky issue about reduced-ness of the cross effects was an actual problem. When I returned to the question in the fall, I realized that switching to the category of simplicial sets should solve the problem and (hopefully!) allow things to generalize. I could have posted the paper to the Arxiv in fall 2016, but I was held back by fear of the mass of imperfections that plagued the project from the start. I continued to worry over it for another semester, and finally posted the preprint after a week of hiking in the mountains of Montana.

I have given many talks on the subject, in various states of feeling fraudulent, sometimes totally confident in the results, and sometimes terrified that someone would see the flaw I was actively trying to mend. Despite my doubts, there was always at least one person stubbornly optimistic the project would work out, and that was Randy, so I owe him for convincing me to keep working; at some point there was a vague threat that if I didn't finish, I would be obligated to name my firstborn child after him.

References

[AC11] Greg Arone and Michael Ching, *Operads and chain rules for the calculus of functors*, Astérisque **338** (2011), vi+158 (English, with English and French summaries).

[AC16] ______, *Cross-effects and the classification of Taylor towers*, Geom. Topol. **20** (2016), no. 3, 1445–1537.

[JM03] B. Johnson and R. McCarthy, *A classification of degree n functors. I*, Cah. Topol. Géom. Différ. Catég. **44** (2003), no. 1, 2–38 (English, with French summary).

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