Enchiridion: Mathematical User's Guides Volume 3 (September 2017)
mathusersguides.com

A user's guide: A monoidal model for Goodwillie derivatives

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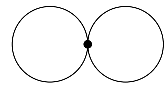
4. Colloquial summary

When my calculus students ask me what I do, I respond that my area is algebraic topology, and they wrinkle their noses and say, "you don't just do harder and harder calculus?"

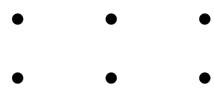
I remind them that one major theme in calculus is that the tangent line is often a very good approximation of a function (near the point of tangency). If any of my students have stumbled in here: that will be on your exam. We can draw the graph of a line $y = mx + b$ fairly easily, and the output of even non-integer inputs are simple to determine. This is a significant motif in calculus. In the first semester, we say, having difficulties with your function? Approximate it with a line! In the second semester, we say, the line was great but maybe we can do better. How about a parabola? We notice this is a little better of an approximation, while not getting too much more difficult to compute, so we try higher-degree polynomials. This leads to infinitely long polynomials, or power series, which can sometimes agree with the original function. The concept of approximating hard things with easier ones is at the heart of a lot of topology.

In my research, I approximate functors. These are a lot like functions, but instead of taking numbers as input, they take objects in a category. In topology, we often study the category of topological spaces. This is a collection of all spaces, along with continuous maps between them; so, for example, there is a circle in this category, and also the map from the circle to itself that wraps around twice. There is a map from the circle to a single point which just squashes it. In my work, I think of each space as having one point that is special. The category of spaces has some cool properties; like numbers, we can add and multiple spaces. Adding two spaces A and B is easy; you just glue together their special points. We denote the sum by $A \vee B$. Below we demonstrate the sum of two circles, where S^1 is the notation for a circle.



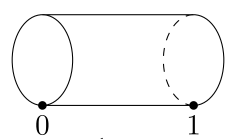
$S^1 \vee S^1$

There are different ways to multiply spaces, but I will only describe one. To multiply space A times space B, at every point in space A, you put a copy of space B. Then you stand back and notice that you could have done it the other way, that is, every point in space B has a copy of space A attached to it. You intuitively know how to do this for basic spaces; it's likely how you learned multiplication. A two point space $A = \bullet\bullet$ times a three point space $B = \bullet\ \bullet\ \bullet$ is a six point space.



$A \times B$

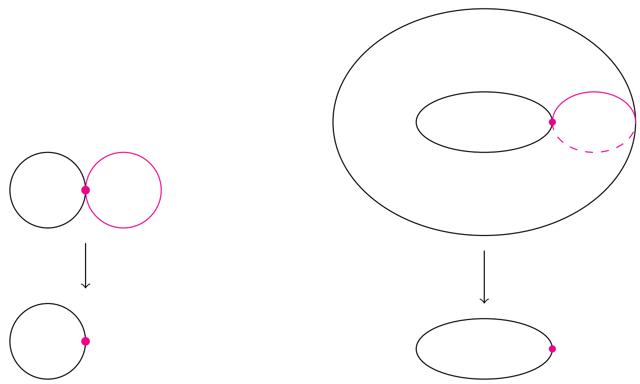
Now if S^1 is a circle and I is the interval $[0, 1]$ on the real line, then $S^1 \times I$ would look like a cylinder.



$S^1 \times I$

I encourage you to take a moment before reading further to think about what $I \times I$ looks like. Then try $S^1 \times S^1$ and a three point space times a filled-in circle. Can you think of two spaces that would multiply to a filled-in 3-dimensional cube? What about a ball?

In the category of spaces with a chosen point, there's also an analogy of subtracting spaces, called the fiber. To demonstrate, let's take two circles which have been added, $S^1 \vee S^1$, then subtract one circle. Notice there is actually a map (or function) from $S^1 \vee S^1$ to S^1 in which we squash one circle down to a point and leave the other alone. Let's say we squash the left one. Then the fiber of $S^1 \vee S^1 \to S^1$ is looking at the inverse image of the point. (We're asking, what does the first space have that the second doesn't?) So the fiber is S^1. Similarly, if we took $S^1 \times S^1$ and looked at the map which squashed one copy down, uniformly all the way around the circle, then the fiber at a chosen point is just the inverse image of that point under the map, which is a copy of S^1. So subtraction of spaces is weird; it doesn't act like ordinary numbers. If $Z - X = Y$, then you know that $Z = X + Y$, but here we have two different examples of Z such that fiber $Z - S^1$ is S^1. I encourage you to try to think of more Z's with this property.



As we've said, functors are a lot like functions, but the domain and range are categories instead of numbers. A function is a way of associating a new number to any number; so $f(x) = x+5$ always adds the number 5 to the input. A functor is like this, but with categories. We can't add 5 to a space, but we can add a space to a space, so we could define a functor from the category of spaces to the category of spaces by $F(X) = X \vee S^1$, which always glues a circle on the chosen point of a space X. There's also a really boring functor, $F(X) = X$, called the identity, and a functor that squashes the entire space to a point, $F(X) = *$, called the trivial functor.

Goodwillie calculus is the approximation of functors with polynomial ones. The definition of polynomial functor is defined in terms of what the functor does to glued together spaces, which means there can be polynomials that don't look like X^n. But a mathematician named Tom Goodwillie defined a degree n polynomial approximation P_nF of a functor F for every n, and he showed that if we subtract $P_{n-1}F$ from P_nF, we always get something like $F' \times X^n$. That's a coefficient space times X multiplied by itself n times, which looks like a standard degree n function. In ordinary calculus, the coefficients on polynomial approximations are given by the higher derivatives of the function, and once you know the derivatives, you know the polynomials because you can just add them up. But because subtraction works differently with spaces, just knowing the derivatives isn't enough to determine the polynomials anymore, so we need to study how the different levels interact. This paper is an effort to understand the derivatives of a functor, their relationship to each other, and how that may influence how the polynomials interact.

References

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