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A user's guide: Categorical models for equivariant classifying spaces

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1. Key insights and central organizing principles

This user's guide is for the paper *Categorical models for equivariant classifying spaces* [**GMM**], which is joint with B. Guillou and P. May. In [**GMM**], we find models for universal equivariant bundles and their classifying spaces as classifying spaces of categories. In this section, after a quick explanation for the setting of the paper, we will give the motivation behind it.

Equivariant bundles are, of course, a generalization of nonequivariant bundles. In this paper, we are only interested in principal (G, Π_G)-bundles $p\colon E \to B$. A principal (G, Π_G)-bundle is nonequivariantly just a principal Π-bundle, but now there are G-actions in sight everywhere, including on the structure group Π, and they need to interact compatibly with the action of the structure group Π on the total space.

Let Π and G be topological groups and suppose that we have an extension of groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{q} G \longrightarrow 1. \tag{1}$$

There is a general theory of equivariant bundles corresponding such extensions (see, for example, [**LM86, May90, May96**]). However, we will only be interested in the case when G acts on Π, the group Γ is the semi-direct product $\Pi \rtimes G$, and the extension is split.

We will refer to bundles corresponding to such extensions as (G, Π_G)-bundles: G is the equivariance group, Π is the structure group, and the subscript in Π_G denotes that G is acting on Π and the bundle corresponds to the split extension given by the semidirect product with respect to this action[1]. If the action of G on Π is trivial, so that $\Gamma = \Pi \times G$, then we omit the subscript G, and refer to such bundles as (G, Π)-bundles[2].

Again, there is a general theory of (G, Π_G)-bundles [**tD69, Las82, LM86, May96**] corresponding to such extensions. The theory is especially familiar when G acts trivially on Π. With $\Pi = O(n)$ or $U(n)$, the trivial action case gives classical equivariant bundle theory and equivariant topological K-theory. The main result of the preexisting theory is that there is a universal principal (G, Π_G)-bundle

$$E(G, \Pi_G) \to E(G, \Pi_G)/\Pi$$

and models for the total space $E(G, \Pi_G)$ and $B(G, \Pi_G) = E(G, \Pi_G)/\Pi$ existed. However, these models were not as classifying spaces of categories.

In [**GMM**], we give models for the total space $E(G, \Pi_G)$ and the classifying space $B(G, \Pi_G)$ of (G, Π_G)-bundles as classifying spaces of categories. The reason why it is important to have such models is two-fold: they are needed in equivariant infinite loop space theory and in equivariant algebraic K-theory. We address how bundle theory comes into the picture for each of these two topics.

1.1. Motivation 1: Equivariant infinite loop space theory. Infinite loop spaces satisfy a recognition principle: they are algebras over E_∞-operads in Top (see [**May72**]). Algebras over an E_∞-operad in $\mathscr{C}at$ are categories whose classifying spaces are, after group completion, infinite loop spaces. The same story carries through equivariantly for a finite group G. Equivariant infinite loop spaces (or infinite loop G-spaces) are G-spaces which have deloopings with respect to all finite dimensional representations of G, so they are zeroth spaces

[1] In order to be consistent with [**GMM**], we do not use the notation from [**May96**] for bundles corresponding to extensions (1). Their notation is (Π, Γ)-bundles, namely the structure group is listed first and the extension group second. In the notation from [**May96**], the bundles we are considering are $(\Pi, \Pi \rtimes G)$-bundles.

[2] The notation for bundles corresponding to extensions with $\Gamma = \Pi \times G$ is consistent with [**May96**], where they adopt the same convention for the trivial action case, and we felt that our notation for the general case better generalizes this.

of genuine G-spectra. Equivariant infinite loop spaces are recognized as algebras over equivariant E_∞-operads in $G\,\mathrm{Top}$ (see [**LMS86**]).

A new development in equivariant infinite loop space theory is defining an E_∞-operad in $G\mathscr{C}at$ such that algebras over it are G-categories whose classifying spaces are, once group completed, infinite loop G-spaces (see [**GM**]). For this it is crucial to have models for equivariant universal bundles as classifying spaces of categories, as we go on to explain.

Nonequivariantly, an E_∞-operad $\mathscr{O}$ in Top has spaces $\mathscr{O}(j) \simeq E\Sigma_j$, namely, universal Σ_j-bundles. An E_∞-operad $\mathscr{O}$ in $\mathscr{C}at$ is defined by the property that the space-level operad $B\mathscr{O}$ with spaces $B\mathscr{O}(j)$ is an E_∞-operad in Top. Let $\widetilde{\Sigma}_j$ be the category with objects the elements of Σ_j and a unique morphism between any two objects. Therefore any object is both initial and terminal, and $\widetilde{\Sigma}_j$ is a contractible category. Also, it has a free Σ_j-action, so $B\widetilde{\Sigma}_j \simeq E\Sigma_j$, thus the categorical operad $\mathscr{O}$ with categories $\mathscr{O}(j) = \widetilde{\Sigma}_j$ is an E_∞-operad. This is also known as the *Barratt-Eccles operad*, and algebras over $\mathscr{O}$ are permutative categories [**May74**].

The definition of an equivariant E_∞-operad $\mathscr{O}_G$ in $G\,\mathrm{Top}$ is in terms of equivariant universal bundles: the spaces $\mathscr{O}_G(j)$ are defined to be universal (G, Σ_j)-bundles, which we denote for now as $E(G, \Sigma_j)$. These are universal principal Σ_j-bundles, with total and base G-spaces, G-equivariant projection map, and commuting actions of G and Σ_j on the total space. We emphasize that here the equivariance group G is acting trivially on the fiber Σ_j. Models for universal equivariant bundles and their classifying spaces are described in [**May96**, VII], for example, but they are not given in terms of classifying spaces of categories.

An E_∞-operad $\mathscr{O}_G$ in $G\mathscr{C}at$ is defined by the property that applying the classifying space functor levelwise yields an E_∞-operad in $G\,\mathrm{Top}$. Thus finding an E_∞-operad in $G\mathscr{C}at$ amounts to finding models for equivariant universal principal (G, Σ_j)-bundles as classifying spaces of G-categories. We summarize this in Table 1 below.

	A nonequivariant E_∞- operad $\mathscr{O}$	An equivariant E_∞- operad $\mathscr{O}_G$
in Top	has spaces universal Σ_j-bundles, i.e., $\mathscr{O}(j) \simeq E\Sigma_j$ **example:** $\mathscr{O}(j) = B\widetilde{\Sigma}_j$	has spaces universal (G, Σ_j)-bundles, i.e., $\mathscr{O}_G(j) \simeq E(G, \Sigma_j)$
in $\mathscr{C}at$	is defined such that $B\mathscr{O}(j) \simeq E\Sigma_j$ **example:** $\mathscr{O}(j) = \widetilde{\Sigma}_j$	is defined such that $B\mathscr{O}_G(j) \simeq E(G, \Sigma_j)$

Table 1: E_∞ operads

From the table, we can see that in order to have a definition of $\mathscr{O}_G$ in $\mathscr{C}at$, for each j, we need a category $\mathscr{O}_G(j)$ whose classifying space is a universal principal bundle $E(G, \Sigma_j)$.

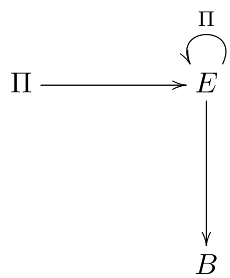
1.2. Motivation 2: Equivariant algebraic K-theory. Quillen's first definition of higher algebraic K-groups was as the homotopy groups of a space $BGL(R)^+$, which turns out to be homotopy equivalent to the basepoint component of the group completion of the topological monoid $B(\coprod_n GL_n(R)) = \coprod_n BGL_n(R)$. Note that this is the topological monoid of classifying spaces of principal $GL_n(R)$-bundles under Whitney sum. Equivariantly, we are unconcerned with any variant of Quillen's original plus construction, but we instead replace the classifying spaces of principal $GL_n(R)$-bundles by classifying spaces of equivariant principal bundles, before group completion.

Note that in contrast to the equivariant bundles considered in the previous section, when G was not acting on Σ_j and we had commuting actions on the total space, now we are assuming that G acts on R, which induces an action on $GL(R)$. The whole point is to take this action into account. The bundles which we are trying to understand are $(G, GL_n(R)_G)$-bundles; they are universal principal $GL_n(R)$ bundles, but they have twisted actions on the total space, i.e., they have an action of the semidirect product $GL_n(R) \rtimes G$ on the total space. The base space is a G-space and the projection map is G-equivariant.

The intuition of defining the equivariant algebraic K-theory space of a G-ring in terms of classifying spaces of $(G, GL_n(R)_G)$-bundles is right, in the sense that we are rigging the spaces to provide an algebra over an E_∞-operad in $G\,$Top that can be fed into an equivariant infinite loop space machine. We refrain to say more about this here, because algebraic K-theory is not really the topic of [**GMM**]; however, the motivation for me was to use these results in my thesis work on equivariant algebraic K-theory. The main result of [**GMM**] is in a sense the starting point of my thesis.

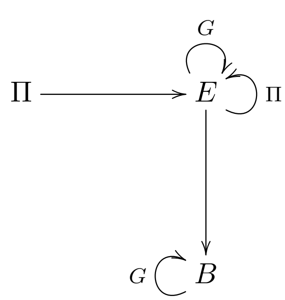
2. Metaphors and imagery

We will start with a remark on notation which was also made in the previous section, but we would like to emphasize it to avoid confusion. Usually when one talks about principal G-bundles, one refers to bundles with structure group G - this means that G acts on the right on the total space such that the action preserves the fibers and is free and transitive on the fibers, which are homeomorphic to G. We will instead use the notation Π for the structure group – this is simply because we will use G for the equivariance group. We can picture a principal Π-bundle as in the following diagram; Π acts on the total space and the fibers are Π. This is so far a nonequivariant principal Π-bundle:

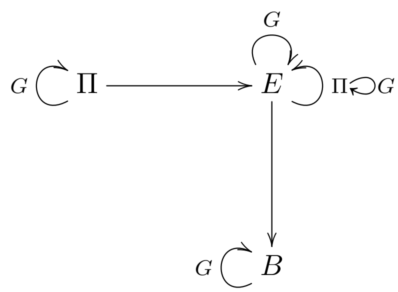


We will introduce the actions of G in two steps, starting with the easier case and then moving on to the more complicated and more general case, so that the reader can slowly build a mental image of where and how G acts in the definition of a G-equivariant principal Π-bundle. Such a bundle will of course, in particular, be a principal Π-bundle.

First, let us think of the "easier" case in which the group G does not act on the structure group Π. In this case a principal G-equivariant Π-bundle, or a principal (G, Π)-bundle, is a principal Π-bundle in which we have a left G-action on the total space which commutes with the Π-action and a G action on the base space such that the projection map is equivariant. With a little bit of algebraic manipulation, it is not hard to see that two commuting actions give rise to an action of the direct product of the two groups, thus on the total space we have an action of $\Pi \times G$. We can picture this by adding G-actions in our previous diagram in the appropriate places -the G-actions do not interact with the Π-action:



The more general and harder case is the one in which the group G acts on the structure group Π. In this case a principal G-equivariant Π-bundle, or a principal (G, Π_G)-bundle, is a principal Π-bundle in which the actions of G and Π on the total space satisfy a *twisted* commutation relation, which translates into an action of the semidirect product (instead of the product) $\Pi \rtimes G$, which again is free when restricted to Π, and the projection map is again G-equivariant. The picture from before just gets a just little more complicated as we add a G-action on the fiber Π and twist the actions of the two groups on the total space:



So far we have tried to build the image of a principal (G, Π_G)-bundle when G acts on the structure group Π. Now, nonequivariantly we know that every principal Π-bundle arises as the pullback of the *universal* principal Π-bundle $E\Pi \to B\Pi$ along some classifying map of the base into $B\Pi$. Namely, if $E \to B$ is a principal Π-bundle, E is the pullback of a diagram

$$\begin{array}{ccc} E & \longrightarrow & E\Pi \\ \downarrow & & \downarrow \\ B & \longrightarrow & B\Pi \end{array}$$

This is also true equivariantly: there exists a universal principal (G, Π_G)-bundle $E\Pi_G \to B\Pi_G := E\Pi_G/\Pi$ so that every principal (G, Π_G)-bundle is the pullback of this bundle along an equivariant classifying map. If $E \to B$ is now a principal (G, Π_G)-bundle, it fits in a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & E\Pi_G \\ \downarrow & & \downarrow \\ B & \longrightarrow & B\Pi_G \end{array}$$

Therefore, in order to understand principal (G, Π_G)-bundles, one needs to understand the principal universal bundle $E\Pi_G \to B\Pi_G$.

The main result of this paper gives a model of the universal principal (G, Π_G) bundle in terms of classifying spaces for categories. We try to give some imagery for the categories that show up.

2.1. G-categories. A G-category $\mathscr{C}$ can be thought of as a functor

$$G \to \mathscr{C}at$$

from G, regarded as a one object category, to the category of categories. Explicitly, the one object of G maps to $\mathscr{C}$, and each morphism of G, namely each element g, maps to an endofunctor of the category $\mathscr{C}$. Functoriality encodes the usual conditions

$$g(hx) = ghx \qquad \text{and} \qquad ex = x$$

for an object or morphism x. In addition, composition needs to be respected:

$$g(f \circ f') = gf \circ gf'$$

for morphisms f, f' in $\mathscr{C}$ since each g is a functor $\mathscr{C} \to \mathscr{C}$.

Thus we can visualize a G-action on a category as permuting both the objects and the morphisms. For any subgroup $H \subseteq G$, the fixed points $\mathscr{C}^H$ are the subcategory formed by those objects and morphisms that are fixed under the action of all the elements of H.

2.2. The category $\widetilde{G}$. One of they key players is the category $\widetilde{G}$, which is defined for any group G to have as objects the elements of G and exactly one morphism between any two objects. This means in particular that any two objects are isomorphic to each other via a unique isomorphism. This is an instance of the more general concept of *chaotic or indiscrete category* defined for any space to have as objects the points in the space and a unique morphism between any two objects.

The category $\widetilde{G}$ is G-isomorphic to the translation category of G –they both have as objects the elements of G on which G acts by translation and since there is a unique morphism between any two objects in both of these categories, the action on objects completely determines the action on morphisms. The only reason why we are saying that $\widetilde{G}$ and the translation category of G are only G-isomorphic and not equal is that the actions are not identical as a result of the choice of labeling of the morphisms. In $\widetilde{G}$, the morphisms are labelled by source and target, namely the unique morphism from g to h is labelled by (h, g), which forces the action to be diagonal on morphisms. (The reversal of source and target is only to make composition more transparent.) In the translation category of G, on the other hand, the morphism labelled (g, h) stands for the morphism starting at h and ending at gh, which forces the action on morphisms to be on the right.

A key feature of the category $\widetilde{G}$ is that its classifying space is contractible: since there is a unique morphism between any two objects in $\widetilde{G}$, any object is initial and terminal. What having an initial or terminal object in a category buys is that upon applying the classifying space construction the resulting space is contractible. Another nice feature of the classifying space construction is that if we apply it to a category with free G-action, the action on the resulting space is also free. Therefore, since the action on $\widetilde{G}$ is free, the action on the classifying space is also free. So $B\widetilde{G}$ is a contractible space with a free G-action, which means that it is equivalent to EG.

Note that $\widetilde{G}$ is *not* equivariantly contractible – that would mean that all the fixed point categories are contractible, but $\mathscr{C}^H$ for $H \neq G$ is empty, so it is not contractible.

2.3. The category $\mathscr{C}at(\widetilde{G}, \mathscr{C})$. Suppose $\mathscr{C}$ is a G-category. Then $\mathscr{C}at(\widetilde{G}, \mathscr{C})$ is the category of *all* functors $\widetilde{G} \to \mathscr{C}$ and natural transformations between these. We emphasized the word "all" because it is crucial that we are taking all functors and natural transformations, not just the equivariant ones, in order to define a G-action on the category $\mathscr{C}at(\widetilde{G}, \mathscr{C})$. The G-action on $\mathscr{C}at(\widetilde{G}, \mathscr{C})$ is by conjugation, so the fixed points $\mathscr{C}at(\widetilde{G}, \mathscr{C})^G$ of this G-category are precisely the equivariant functors and natural transformations because those are the ones that are invariant under conjugation.

The objects of $\mathscr{C}at(\widetilde{G}, \mathscr{C})$ are not hard to visualize: a functor $\widetilde{G} \to \mathscr{C}$ is just a diagram of objects in $\mathscr{C}$ indexed over the elements of G together with a unique isomorphism between any two of them.

2.4. The model for equivariant bundles and a reality check. The model that we find in the paper for the universal principal (G, Π_G)-bundle when G is finite or discrete, but Π is allowed to be compact Lie, is

$$B\mathscr{C}at(\widetilde{G}, \tilde{\Pi}) \to B\mathscr{C}at(\widetilde{G}, \Pi).$$

Maybe this is not that easy to visualize, but let us just do a reality check: If we forget the G-actions altogether, this is supposed to be a universal Π-bundle.

Since $\widetilde{G}$ is a contractible category nonequivariantly, we have nonequivariant equivalences of categories

$$\mathscr{C}at(\widetilde{G}, \tilde{\Pi}) \simeq \tilde{\Pi} \qquad \text{and} \qquad \mathscr{C}at(\widetilde{G}, \Pi) \simeq \Pi,$$

and since $B\tilde{\Pi} \simeq E\Pi$, nonequivariantly this bundle is equivalent to

$$E\Pi \to B\Pi.$$

The last image we want the reader to keep in mind is that a universal principal (G, Π_G)-bundle is nonequivariantly the universal Π bundle $E\Pi \to B\Pi$, but with some funky G-actions everywhere. Replacing the categories $\tilde{\Pi}$ and Π by the nonequivariantly equivalent categories $\mathscr{C}at(\widetilde{G}, \tilde{\Pi})$ and $\mathscr{C}at(\widetilde{G}, \Pi)$ builds in "the right equivariant homotopy type." We haven't discussed in this users guide what this "right" equivariant homotopy type is, but let us just say here that there is an explicit characterization of the $\Pi \rtimes G$ homotopy type of the total space in a universal (G, Π_G)-bundle that one can check, just like there is a characterization of the Π-equivariant homotopy type of $E\Pi$: the fixed point space for the trivial subgroup $E\Pi^e = E\Pi$ is contractible and all the other fixed point subspaces $E\Pi^H$ for subgroups $H \neq e$ are empty.

3. Story of the development

The paper [**GMM**] was the starting point of my thesis work. In Section 1 of this user's guide, I gave two motivations for the main result of the paper: one was that it motivates the definition of equivariant algebraic K-theory, which is what I was beginning to work on, and the second was that it provides a model for an E_∞-operad in G-categories, which in turn lets one develop operadic equivariant infinite loop space theory, and this was what my co-authors Bertrand Guillou and Peter May were working on. This was a fortunate collision of interests which resulted in jointly writing this paper, and a lot of subsequent joint work that naturally followed.

By the end of my second year of graduate school, my interest was totally piqued by algebraic K-theory because of its connections to numbers theory, which was what I initially wanted to study before being lured by Peter May into the world of algebraic topology. I spent a long time learning about algebraic K-theory for my topic exam, and I was thirsty for more. On the other hand, Peter May enticed me with the wonderful world of equivariant homotopy theory, and that is how the general idea of equivariant algebraic K-theory ossified into a thesis project.

Now "equivariant algebraic K-theory" had to be made sense out of, and the first problem my adviser, Peter May, suggested was to prove an "equivariant plus=Q" theorem. The classical "plus=Q" theorem is a deep result of Quillen stating that two very different definitions that he gave of algebraic K-theory of a ring agree, namely his "plus construction" and his more general "Q-construction" for exact categories, when specified to the category of finitely generated projective modules over the ring.

I had no idea at that point what an equivariant version of this would be. Peter directed me to the relevant literature, which consisted of a paper by him, Fiedorowicz and Haushild from the 1980's in which they study a space-level version of equivariant algebraic K-theory for rings with trivial G-action [**FHM82**] and some old papers by Dress and Kuku in which they give and study a definition of equivariant algebraic K-groups for exact categories with trivial G-action [**DK82**]. I formulated a clear task for myself, which was to reconcile these definitions. I was really thrilled when I worked this out, because it was the first problem I had gotten. When I excitedly told Peter I had figured it out, he was very pleased but he told me that the work has just begun. And so it was.

The definitions I had reconciled only worked for trivial G-action on the input ring. My next goal was to generalize these definitions to rings with nontrivial action. The equivariant "plus" construction definition from [**FHM82**] is in terms of classifying spaces of equivariant $GL_n(R)$-bundles, but the kind where the equivariance group G does not act on the fiber $GL(R)$. Even in that setting the model used for these classifying spaces was very unwieldy. One of the main ingredients in my reconciliation of their definition with Dress and Kuku's definition was to use a different model for the classifying space of a $(G, \Pi \times G)$-bundle, one that was the classifying space of a category, and which I had learned from a working draft of Bertrand Guillou and Peter May on equivariant infinite loop space theory. Thus my main task narrowed down to bumping up their model to the case of twisted actions – the different complexity levels of the equivariance were described in the previous section.

After working on this for a while during the summer of 2011, I showed that the categorical model for the total space of equivariant universal principal bundles from Bert and Peter's note generalizes to the case when G acts on the fiber group. The main idea was to replace group homomorphisms that were showing up with crossed homomorphisms. However, going into this project I did not know what a crossed homomorphism was, and the breakthrough definitely happened when I came across the definition of a crossed homomorphism and I realized that a lot of things I was encountering were crossed homomorphisms, which were central to understanding the twisted actions.

The classifying space for an equivariant principal bundle is the quotient of the total space by the structure (fiber) group. In order to obtain the desired classifying space that I was actually using for comparing the "plus" and "Q" definitions even in the case of trivial G-action, one has to show that the quotient of the total space with the structure group satisfies two commutations that were not obvious. However, neither Peter and Bert in their note, nor I had written a proof of the commutations that would give the right classifying space, and without them my comparison result of "plus=Q" in the case of trivial G-action and its generalization to nontrivial G-action would fall apart.

That was the point when Peter suggested all three of us jointly work to finish off this project on categorical models of classifying spaces of equivariant bundles and make it a separate paper since the results are of independent interest. The work was laid out for us at this point, and we carried out the verifications that we needed. Then we also computed the fixed point spaces of the classifying spaces of these equivariant bundles in terms of fixed point categories, and a very pleasant surprise arose when during discussions with two number theory postdocs at UChicago at the time, Matthew Morrow and Liang Xiao, I have learned about nonabelian cohomology, and realized that first nonabelian cohomology sets $H^1(G, GL_n(E))$ appeared as the isomorphism classes of objects of these fixed point categories when G is the Galois group of a finite field extension E/F. This was crucial when I later on had to use the result from our paper in my thesis on equivariant algebraic K-theory.

This paper has been in many ways a starting point. It was not only the starting point of my work on equivariant algebraic K-theory, but it was also the beginning of a long collaboration with my adviser Peter May and Bertrand Guillou that is still ongoing, and that Angelica Osorno who was then a postdoc at UChicago also became a part of. I have been very lucky to learn from all of them over the years.

4. Colloquial summary

In topology we study spaces, which are sets of points with a notion of nearness. Some examples of spaces are a point, a finite number of points in the plane, a circle, a sphere, a higher dimensional sphere, Euclidean space, a doughnut shape - these are all easy to visualize. From spaces we know, we can create new spaces: For example, if we take two circles and take their "product," which is the space that we obtain by sliding one circle around the other one, we obtain the doughnut shape. Another construction that gives this same space is taking a rectangle, glueing the top edge to the bottom edge, and glueing the left edge to the right edge. Just using basic constructions like these, we can get bigger and bigger spaces. Most spaces that a topologist considers are infinite-dimensional and very complicated, but they are built out of basic building blocks called cells, which are glued together according to a given recipe.

In the eyes of a topologist two spaces are deemed equivalent if one can continuously be transformed into the other without breaking it apart. For instance, a circle and a square are equivalent because one can be continuously deformed into the other. However, a sphere is, for example, not equivalent to a circle, because there is no way to deform it into a circle without breaking a hole, or rather two holes in it. The Euclidean plane is equivalent to a disk centered around the origin, you can imagine shrinking the plane from all directions into a disk. Now you can further shrink the disk into its midpoint. So the plane is equivalent to the disk, which in turn is equivalent to a point. Through the lens of algebraic topology these spaces are the same. But clearly the entire Euclidean plane is infinite and much more unwieldy than the disk. The disk still has infinitely many points, so it is still much more complicated than just the point. So from all these models for the same topological space up to equivalence, the point is clearly the easiest one to work with.

There are some spaces in algebraic topology that are only defined up to equivalence: we know that there exists a space satisfying a property X and all spaces satisfying property X are equivalent. An example of a space defined up to equivalence is the "classifying space of a bundle with certain fiber." A bundle consists of two spaces, a base space and a total space, and a projection map from the total space to the base space such that the preimage of each point in the base under this projection map, called the fiber of the bundle, has the some nice structure. What it means for a space to be a classifying space for a bundle with a certain fiber is that every other bundle with that same fiber corresponds in a precise way to a map from its base space to the classifying space. Given a bundle with a certain fiber, its classifying space is defined up to equivalence: we know there exists a space with the property we just described and also that any other space that has the described classifying property is equivalent to it. Therefore, when it comes to classifying spaces of bundles, one needs to pick and work with one of many possible models for the same space.

In this paper we consider "equivariant bundles," which are bundles that have additional structure on them. Very roughly, they are bundles with the additional data of their symmetries. The classifying spaces for equivariant bundles also have extra data that encodes symmetries. There were previously known models for such classifying spaces, but they were huge and unwieldy spaces. As we have seen above, there can be many models of the same space, which are all equivalent, some easier to understand than others. In the paper we find models for classifying spaces of equivariant bundles that are more convenient to analyze for certain applications than the previously known models. The equivalent models that we find arise from categories - a category is a collection of objects and directed relations between them, called morphisms, or simply maps. The spaces built out of categories are infinite dimensional, but they follow a very precise recipe dictated by the objects and relations between them, and can therefore be easier to understand than other spaces. Of course, it all depends on the complexity of the category in question, but the models that we find come from uncomplicated categories with very simple relations between the objects.

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