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A user's guide: The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic p-groups

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1. Key ideas and central organizing principles

1.1. Background. In stable homotopy theory, we can regard Eilenberg-MacLane spectra as the fundamental "building blocks" of other spectra. This notion is embodied by the Postnikov tower which encapsulates how a spectrum is constructed from its individual homotopy groups. The key characteristic then of this particular filtration is that the fibers are indeed Eilenberg-MacLane spectra. However, in equivariant stable homotopy theory, that is, in the context of spectra with a group action, the role of Eilenberg-MacLane spectra is not so simple.

To begin looking into the equivariant setting, we must note that the homotopy "groups" of genuine equivariant spectra are not merely groups. These homotopy groups for a G-spectrum X come from the fixed points of equivariant maps from S^n to X. However, to get the full picture, instead of merely considering the G-fixed points, we must consider the H-fixed points for all subgroups H of G. The homotopy groups of X^H for all H also have maps between them and all this data fits together to form a *Mackey functor*.

One might expect then that the proper way to extend the Postinikov tower to the equivariant setting would be to construct a tower whose fibers were all Eilenberg-MacLane spectra associated to Mackey functors. However, this type of filtration does not end up being the most natural choice for classic equivariant spectra. What we really want is a filtration that uses representation suspensions of Eilenberg-MacLane spectra; this is the *slice filtration.* This filtration for a G-spectrum X is formed in a similar fashion to the Postnikov tower but instead of killing maps (i.e. formally inverting them up to homotopy) from ordinary spheres to X, we kill maps from so-called *slice cells.* The result is that while we still get a tower whose limit is equivalent to X and whose colimit is contractible, the fibers are often more complicated G-spectra that we refer to as *slices.*

In [**HHR09**] the slice filtration was shown to filter the complex cobordism spectrum in a nice way. However, when applied to other spectra, we can get some more complicated results. What becomes mysterious in considering a slice tower rather than a Postnikov tower is that often the homotopy of each layer of the slice filtration differs from the next in more than one dimension. Additionally, if we have an object that is n-slice, or capable of being the fiber in the nth layer of a tower, and we suspend it, it may not be $(n+1)$-slice. Thus, the slice tower does not commute with taking integer suspensions. To get a better idea of how suspension and the slice filtration interact, it would be nice to know the slice towers for all suspensions of simple objects. As a start, in [**Yar15**] we wanted the following:

Goal: Determine the slice tower for all positive integer suspensions of $H\underline{\mathbb{Z}}$, the Eilenberg-MacLane spectrum associated to the constant G-Mackey functor where G is a cyclic p-group for p an odd prime denoted by C_{p^k}.

1.2. Discussion of the main result.

In order to describe the tower fully, we must determine each stage (P^nX) of the tower, all slices of the tower (P^n_nX), and show that these pieces fit into successive fiber sequences

$$\begin{array}{ccc} P^n_nX & \longrightarrow & P^nX \\ & & \downarrow \\ & & P^{n-1}X \end{array}$$

Our main result in [**Yar15**, Section 4], essentially gives us a blueprint for constructing the slice towers for our selected spectra. We first state the exact form of all nontrivial slices and the dimensions in which they occur. From this information we know the exact dimensions in which the stages of the tower change. We can then determine the form of the spectra that make up each stage and confirm that such spectra do fit into an appropriate tower as the pieces of successive fiber sequences. The following are the key ideas that together form the main result.

Key Idea 1.1. *The nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ are of the form $S^{V(a,b)} \wedge H\underline{B}_{i,j}$.*

$V_{(a,b)}$ is a C_{p^k}-representation obtained from removing a number of irreducible subrepresentations from copies of ρ_G, the regular representation of C_{p^k}. The definition of this representation is given exactly in Definition 3.1. $\underline{B}_{i,j}$ is a Mackey functor obtained by taking quotients of maps from $\mathbb{Z}(i,j)$ to $\mathbb{Z}$ where $\mathbb{Z}(i,j)$ is a slight alteration of the constant Mackey functor. A precise formula is given in Definition 2.3. That such spectra are slices of a particular dimension is shown in the proof of Theorem 3.2.

KEY IDEA 1.2. *These nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ occur in dimensions $mp^i - 1$ where where $1 \leq i \leq k$ and m is an integer of the same parity as n that occurs in a particular finite range.*

This means that the successive stages of the tower will only change in particular dimensions that are one less that a multiple of a power of p. These dimensions are simply the dimensions of the representations $V_{(a,b)}$ given in Key Idea 1.1. Additionally, we note that the ranges of i and m are finite and thus the tower itself is finite. This means that we actually have $S^n \wedge H\underline{\mathbb{Z}}$ in the top layer and eventually we get to the trivial spectrum. The remaining stages are briefly described in the next Key Idea.

KEY IDEA 1.3. *The nontrivial spectra that make up the stages of the tower, that is $P^i S^n \wedge H\underline{\mathbb{Z}}$, are all of the form $S^V \wedge H\underline{\mathbb{Z}}$ where V is a C_{p^k}-representation of dimension n.*

The towers for $S^n \wedge H\underline{\mathbb{Z}}$ are finite and thus at the top of the tower of course we must have $S^n \wedge H\underline{\mathbb{Z}}$ itself. As we work our way down the tower, we see that at level in which the tower changes, a 2-dimensional representation or two trivial representations will be swapped out for another 2-dimensional representation with fewer G-fixed points.

KEY IDEA 1.4. *The slices are the fibers of maps between successive layers of the spectra described in Key Idea 1.3.*

That is, the slices and stages do in fact fit into fiber sequences. These fiber sequences form the tower beginning at $S^n \wedge H\underline{\mathbb{Z}}$ and terminating at the trivial spectrum. There are patterns that arise in all this data and this shall be discussed more thoroughly in the next section.

1.3. Discussion of the proof. To prove that the data given in the Key Ideas above in fact gives us the slice tower we need to show that the limit is equivalent to $X = S^n \wedge H\underline{\mathbb{Z}}$, the colimit is contractible, every i-dimensional fiber is in fact an i-slice, and these slices and successive stages we've determined do indeed form fiber sequences

$$\begin{array}{ccc} P^i_i X & \longrightarrow & P^i X \\ & & \downarrow \\ & & P^{i-1} X \end{array}$$

As previously mentioned, the towers presented above are finite, so the only interesting work amounts to showing each fiber is an i-slice and that appropriate fiber sequences may be formed.

To show that a spectrum is an i-slice, one must show that it is $\leq i$ and $\geq i$ as defined in [**HHR09**] or [**Hil12**]. As stated in Key Idea 1.1 the candidates for the slices of our tower were of the form $S^{V_{(a,b)}} \wedge H\underline{B}_{i,j}$. In general, to show a spectrum of the form $S^V \wedge H\underline{M}$ is $\geq \dim(V)$, by [**Hil12**, Theorem 3.7], we need only show that $V \subset (m\rho_G - 1)$ for some integer m where $V^G = (m\rho_G - 1)^G$. Additionally, to show $S^V \wedge H\underline{M} \leq \dim(V)$, we can induct on the subgroups of G. By our inductive hypothesis, that is that $S^{|V|} \wedge H\underline{M}$ with a trivial action is $\leq |V|$, and Spanier-Whitehead duality, it will be sufficient to show that $[S^{-\epsilon}, S^{V - t\rho} \wedge H\underline{M}] = 0$ for $tp^k - \epsilon > \dim(V)$ and $\epsilon = 0, 1$. To do so, we really only need to show that the related homology in dimension $-\epsilon$ is trivial and thus are able to make arguments using chain complexes. This method of showing particular spectra are slice is summarized in [**Yar15**, Theorem 5.9].

To show that we have appropriate fiber sequences, we rely heavily on the fact that working p-locally means that our spectra $S^V \wedge H\underline{B}_{i,j}$ and $S^{V'} \wedge H\underline{B}_{i,j}$ will still be equivalent even when V and V' differ by certain subrepresentations of ρ_G. In particular, $H\underline{B}_{i,j} \simeq S^{\lambda_l} \wedge H\underline{B}_{i,j}$ for $l \leq j$ where $\lambda_l : C_{p^k} \to S^1$ is the composition of the inclusion of the p^kth roots of unity with a degree p^ith map on S^1. Essentially, the representation λ_l has too few fixed points to be "seen" by $\underline{B}_{i,j}$ since $\underline{B}_{i,j}$ is trivial on subgroups C_{p^l} for $l \leq j$. Thus, while our descriptions of the slices and layers of the tower may not seem to fit exactly into the correct sequences, they are all equivalent to spectra that do.

2. Metaphors and imagery

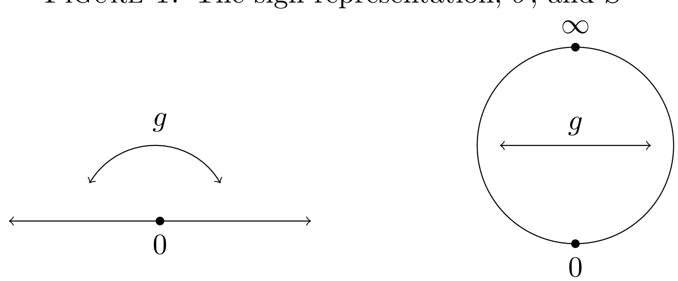
2.1. Equivariant stable homotopy theory. Before discussing the imagery I use concerning the specific results of the paper [**Yar15**] or even the slice filtration itself, it will be best to briefly describe how I think about the basic objects used, G-spectra and Mackey functors.

2.1.1. *G-spaces and representation spheres.* As the story of G-spectra is most fully told by relying on the collection of G-spaces called representation spheres, it is most fitting to begin here. A G-space is a topological space with an action of a group G. What we imagine is a collection of points that become permuted in some way when applying elements from the group. The action induced by any given element may be *trivial* in which no points are moved or it may be *free* in which no points are fixed. An important collection of points in the equivariant context is the collection of fixed points for a given subgroup H of G. In reality, we actually need to keep track of the fixed points for all subgroups H of G and thus, one consideration we always carry along with us is the structure of the group lattice.

One type of G-space that is integral to the equivariant stable field is a representation sphere S^V. What we mean by this notation is the one point compactification of the space associated to the representation V. Such a vector space may be thought of as a G-space with action induced in the obvious fashion. That is, each element of G acts on the space as the associated element from the general linear group. Thus, we imagine taking a space $\mathbb{R}^n$ with permuted points and gathering everything at infinity to one point while carrying that group action along. Forming the sphere in this way forces the points at 0 and ∞ to always be fixed. The following examples are common ones that I like to keep in mind.

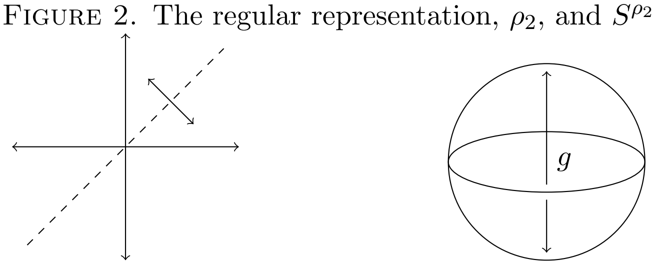
EXAMPLE 2.1. Consider the 1-dimensional sign representation $\sigma : C_2 \to GL(\mathbb{R})$. The associated representation space is depicted on the left in Figure 1. When $\mathbb{R}$ is thought of as a C_2-space with an action induced by σ, 0 is always fixed by the C_2-action and the action of the nontrivial element g swaps the positive and negative points. Its associated representation sphere is depicted on the right.

FIGURE 1. The sign representation, σ, and S^σ



EXAMPLE 2.2. Then the regular representation of C_2 has a decomposition $\rho_2 = 1 + \sigma$. ρ_2 is then a 2-dimensional representation with its associated space shown in Figure 2 on the left. As a C_2-space, the action of the trivial element flips the x- and y-coordinates, fixing the diagonal. Its associated representation sphere is depicted on the right. We can see then that the equator (coming from the diagonal) is always fixed by the action of C_2 and the upper and lower hemispheres are swapped when acted on by the nontrivial element from C_2.

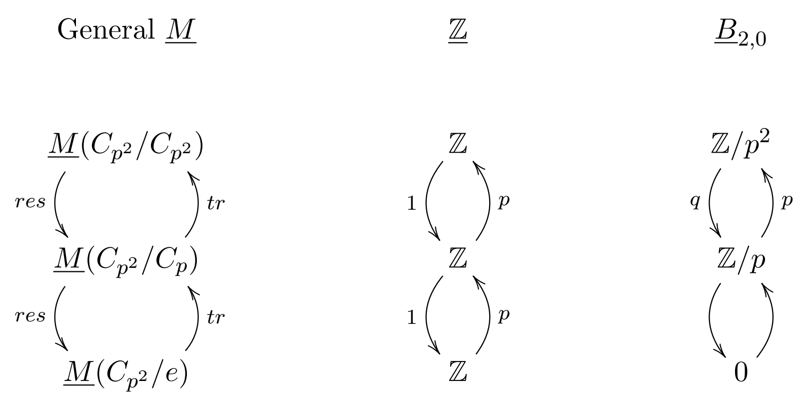
FIGURE 2. The regular representation, ρ_2, and S^{ρ_2}



2.1.2. *G-spectra.* In the context of stable homotopy theory, one may think of a spectrum as a sequence of spaces arranged in an ordered infinite line with structure maps running between them. When considering G-spectra, we must first imagine that each space used in forming the spectrum is a G-space. To picture "naive" G-spectra one need only imagine a similarly infinite line of G-spaces. However, "genuine" G-spectra, those considered in the paper, are a bit more complicated. The G-spaces that fit together to form a spectrum are instead indexed on repesentations of G. There are still infinitely many but they are not arranged linearly. Rather, we could think of the G-spaces arranged in a sort of directed lattice whose structure is determined by the collection of G-representations we are indexing on. Two G-spaces, say X_V and X_W, are connected in the lattice if $V \subset W$. Thus, not only do the spaces that form the spectrum have an action of G but we must also remember that the way they are arranged carries information from the group G as well.

2.1.3. *Mackey functors.* When computing the homotopy "groups" of a G-spectrum, we actually get a collection of groups that fit together to form an object called a *Mackey functor*. A Mackey functor, $\underline{M}$, can be thought of as a bifuntor from the orbits of G to abelian groups. It is also useful to think of a Mackey functor as a lattice but this time, the lattice structure is determined by the group lattice of G. Additionally, as it is a bifunctor, each edge in the lattice has two directions. In [**Yar15**], we only consider G to be a cyclic p-group, and thus all G-Mackey functors in this context can be thought of as "ladders"; the groups $\underline{M}(G/H)$ are the rungs and the contravariant (restriction) and covariant (transfer) morphisms form the sides. See Figure 3 for examples of Mackey functor diagrams. Note that $\underline{\mathbb{Z}}$ is the constant Mackey functor and $\underline{B}_{2,0}$ is a particular case of a Mackey functor defined in [**HHR15**] and used extensively in [**Yar15**].

FIGURE 3. Examples of C_2-Mackey functors



Picturing a Mackey functor in this way is especially useful for performing computations such as homology computations with chain complexes of Mackey functors. In such work, one can then picture a sort of commutative diagram of lattices. Additionally, this imagery impresses upon us the important role that subgroups play in the equivariant realm as each subgroup corresponds to a different point in the lattice or "rung" on the ladder.

2.2. The slice filtration. Most often, the way I imagine the slice filtration is set against the backdrop of the Postnikov tower. The Postnikov tower builds a spectrum one homotopy group at a time and thus one can imagine the homotopy groups as neatly stacked blocks. On the other hand, the slice tower conjures an image of smearing out the homotopy groups of a G-spectrum.

Below, we see the homotopy groups of P_nX and $P_{n-1}X$ in the Postnikov tower:

Dimension	P_nX	$P_{n-1}X$
	$\dots$	$\dots$
$n+2$	0	0
$n+1$	0	0
n	$\pi_n(X)$	0
$n-1$	$\pi_{n-1}(X)$	$\pi_{n-1}(X)$
$n-2$	$\pi_{n-2}(X)$	$\pi_{n-2}(X)$
$\dots$	$\dots$	$\dots$

From this, it is easy to see that the fiber of the map $P_nX \to P_{n-1}X$ has its only nontrivial homotopy group in dimension n, namely, $\pi_n(X)$.

If we consider the layers of the slice tower P_nX in terms of stacked integer-graded homotopy groups, we do not get such a straightforward picture. This is because each homotopy group in dimension n and below need not be the same as the homotopy groups of X. For example, consider P_0X and $P_{-1}X$ shown below:

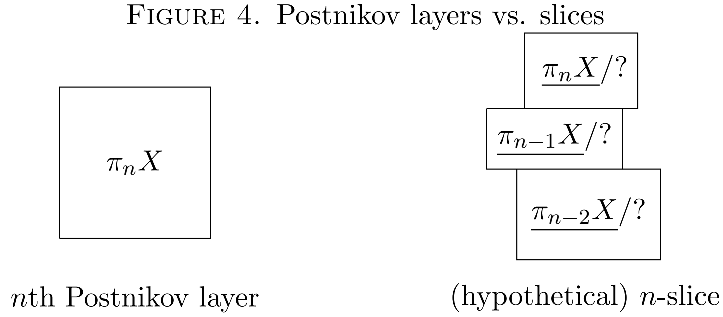
Dimension	P_0X	$P_{-1}X$
	$\dots$	$\dots$
1	0	0
0	$\underline{\pi_0}(X)/*$	0
-1	$\underline{\pi_{-1}}(X)$	$\underline{\pi_{-1}}(X)$
$\dots$	$\dots$	$\dots$

The Mackey functor $\underline{\pi_0}(P_0X)$ is really a quotient of $\underline{\pi_0}(X)$. This is because some of the slice cells we kill maps from in this instance are not as connected as their underlying sphere. In particular, S^σ is not 0-connected even though the underlying space, S^1 is.

We pause to note that there is actually a nice description of 0-slices due to Hill in [**Hil12**] and they are in fact Eilenberg-MacLane spectra. However, for larger slices we do not have a strong understanding. We do not know the homotopy groups in stages above the zeroth in general. In fact, the layers are often so jumbled that the fibers need not be Eilenberg-MacLane spectra and moreover, are often difficult to determine even in individual cases.

We can still think of moving up the slice tower as building the G-spectrum using homotopy information, it's just that we are not adding one integer graded homotopy group at a time. In constructing the nth stage of the slice tower, homotopy groups in dimension n and below may be altered. Thus, we often see G-spectra with many nontrivial homotopy groups as slices. So while one might imagine the "building blocks" in the context of the Postnikov tower as uniform cubes, the building blocks when viewed through the lens of the slice tower can be objects of a variety of sizes and shapes. Figure 4 gives a general visual idea of the comparison between a single Postnikov building block and a single slice building block.

FIGURE 4. Postnikov layers vs. slices



It is important to keep in mind that what is depicted in Figure 4 is only a vague possibility of what we might think about an n-slice. While it is true that we won't ever see any part of an n-slice building block above dimension n, we don't know in general how many lower dimensions the block might touch. We also don't really know the exact "shape" only that it makes sense to think of parts of the block as "smaller" than the homotopy "cubes" since they arise from quotients of the Postnikov layers.

2.3. Concerning main result. Now, how should one think about the specific slice towers for C_{p^k}-spectra of the form $S^n \wedge H\underline{\mathbb{Z}}$? While we might often describe building the tower from the bottom up, I find it useful to imagine beginning at the top and working down when trying to picturing exactly what the slice filtration does to the given spectrum. In the particular case considered in [**Yar15**], every nontrivial sequence in the tower looks like

$$\begin{array}{ccc} S^V \wedge H\underline{M} & \longrightarrow & S^W \wedge H\underline{\mathbb{Z}} \\ & & \downarrow \\ & & S^{W'} \wedge H\underline{\mathbb{Z}} \end{array}$$

where $\underline{M}$ is some C_{p^k}-Mackey functor and V, W, and W' are C_{p^k}-representations, W and W' having dimension n. In this depiction, we mean that $P_n X \simeq S^W \wedge H\underline{\mathbb{Z}}$, $P_{n-1}X \simeq S^{W'} \wedge H\underline{\mathbb{Z}}$, and the fiber of the map, or slice, is $P_n^n X \simeq S^V \wedge H\underline{M}$. Now the question is, what exactly does this mean?

This is the way I "read" such a tower: the slice $S^V \wedge H\underline{M}$ encodes the information that tells us how the stages of the tower are changing. In particular, it is the Mackey functor $\underline{M}$ that determines this; the representation V is essentially forced by how the tower is changing. Furthermore, as mentioned above in Topic 1, these representations can be written in many equivalent ways so it doesn't really make sense to think of V as static. Each Mackey functor $\underline{M}$ is of a certain type (really $\underline{M} = \underline{B}_{i,j}$, see Figure 3 for an example) and I think of them being color coded to tell us how W changes into W'. Each different color prescribes a particular change in the tower. Consider the following example:

Example 2.3. Here is a (modified) portion of the slice tower for $S^{10} \wedge H\underline{\mathbb{Z}}$ with $G = C_9$:

$$\begin{array}{ccc} S^V \wedge H\underline{green} & \longrightarrow & S^{4+2\lambda_1+\lambda} \wedge H\underline{\mathbb{Z}} \\ & & \downarrow \\ S^{V'} \wedge H\underline{red} & \longrightarrow & S^{2+2\lambda_1+2\lambda} \wedge H\underline{\mathbb{Z}} \\ & & \downarrow \\ & & S^{2+\lambda_1+3\lambda} \wedge H\underline{\mathbb{Z}} \end{array}$$

The "green" Mackey functor invokes a change of two trivial representations for one λ while the "red" Mackey functor swaps one λ_1 for one λ. Every λ representation is 2-dimensional so the underlying dimension is preserved. The order in which we write the subrepresentations (trivial, λ_1, λ) lists them from the most fixed point to least fixed points under the group action.

More generally, the change from W to W' is always a 2-dimensional subrepresentation of the regular representation for C_{p^k} being swapped out for another 2-dimensional subrepresentation that has fewer fixed points. When looking at each tower from top to bottom, they seem to swap out as many representations as possible with a large number of fixed points first before swapping out representations with fewer fixed points. In particular, each one of the towers begins with a sequence of changing trivial representations to λ-type representations. Naturally, one would like to know why this is the case. In order to better understand this, one must look instead at the pattern of Mackey functor types as these determine the changes in representations.

The various "colors" of $\underline{M}$ (or types of $\underline{B}_{i,j}$) follow a pattern that relies on the p-adic valuation of n for each $(n-1)$-slice. Figures 5 and 6 give the order the Mackey functor types for the nontrivial slices in the given towers for $G = C_9$. At the top of each we see the same type of functors until the power of $p = 3$ decreases. Then we can see that anytime the slice dimension $n-1$ has a larger p-adic valuation for n we have a different type of functor. Why? Consider, for example, the slice in dimension $12(3) - 1$. We could also write this as $4(3)^2 - 1$ and thus might expect that the functor type is more similar to those slices in dimensions $m(p)^2 - 1$ at the top. This is indeed the case and furthermore is the reason why I think of these as green (closer to blue) rather than red.

FIGURE 5. $S^7 \wedge H\underline{\mathbb{Z}}$

Slice Dimension	Mackey functor type
$5(3)^2 - 1$	blue
$3(3)^2 - 1$	blue
$5(3) - 1$	red
$3(3) - 1$	green

Another property to note is that in each of the towers depicted the red/green pattern that follows the blue slices is the same. In particular, one red slice appears first and each tower ends in green. This is due to the fact that the difference between the suspensions is 9, exactly the size of the group we are considering. We could guess (and we'd be right!) that the tower for $S^{25} \wedge H\underline{\mathbb{Z}}$ would also begin with a number of blue slices, then one red, then a longer p-adic pattern of greens and reds, ending with green. The reason behind this property is intrinsically tied to the interplay between the slice tower and suspensions by regular representation spheres; the dimension of such spheres is of course the order of G. The exact relationship between the slice tower and suspensions is discussed in Topic 3 below.

One last note regarding these patterns is that the number of color block types corresponds to the power of p in the group G. When $G = C_p$, we see only one type of Mackey functor. When $G = C_{p^2}$ as in the tables above, we see two: blue and red/green. When $G = C_{p^3}$ we see three (e.g. blue, red/green, purple/turquoise/white). Additionally, the size of the blocks in each tower is the same (except in the outlier case where the n in $S^n \wedge H\underline{\mathbb{Z}}$ is a multiple of p). This property is a result of the relationship between the slice tower and the notion of

FIGURE 6. $S^{16} \wedge H\underline{\mathbb{Z}}$

Slice Dimension	Mackey functor type
$14(3)^2 - 1$	blue
$12(3)^2 - 1$	blue
$10(3)^2 - 1$	blue
$8(3)^2 - 1$	blue
$6(3)^2 - 1$	blue
$14(3) - 1$	red
$12(3) - 1$	green
$10(3) - 1$	red
$8(3) - 1$	red
$6(3) - 1$	green

restricting to subgroups in the equivariant setting. This relationship is further discussed in Topic 3.

3. Story of the development

3.1. The background. In this section, we look at a brief history behind the problem of determining particular slice towers. Hill, Hopkins, and Ravenel were the first to fully develop and use the notion of the slice filtration as an expansion on the work of Dugger in [**Dug05**]. In [**HHR09**] they presented a formal definition of the filtration and used the particular slices of spectra built out of the spectrum MU in their solution to the Kervaire invariant one problem. While the towers of such spectra were determined rather straightforwardly, the towers of other, even seemingly simpler spectra, can be quite a bit more complicated. Additionally, there is still much unknown about the way the slice filtration filters spectra in general. Thus, one would like to know towers for a variety of spectra to get a better handle on what exactly the slice filtration does to G-spectra.

There are two aspects to consider when choosing the spectra whose towers we will compute. One is the group whose action we are considering. Beginning with cyclic p-groups is a logical place to start as the associated Mackey functors are relatively simple since all subgroups are nested. The other is of course the spectrum itself and suspensions of it by representation spheres. A natural place to begin on this front could be Eilenberg-MacLane spectra and their suspensions. $H\underline{\mathbb{Z}}$ is a classic choice and Hill, Hopkins, and Ravenel express in [**HHR15**] a goal of determining the slice towers of suspensions of $H\underline{\mathbb{Z}}$ by virtual representation spheres. In [**HHR15**], they compute the towers for particular suspensions by subrepresentations of the regular representation associated to cyclic p-groups. In a sense, the paper [**Yar15**] is a complementary one in that it determines the towers for suspensions of $H\underline{\mathbb{Z}}$ by trivial representations.

3.2. The process. While much of the early work on this project was essential to determining the final result, these computations play no part in the actual proof and thus all discussions concerning this part of the process were omitted from the paper. Here, I will lay out the early ideas and useful notions that ultimately lead to the main result.

In beginning to compute the slice tower for any given G-spectrum, there are a few results concerning the slice filtration in general that give a good starting point. One such result, given as Corollary 2.12 in [**Hil12**], provides the form of the (-1)-layer of any slice tower.

USEFUL RESULT 3.1. *The* (-1)*-layer of any* G*-spectrum* X *is the* (-1)*-Postnikov layer:*

$$P_{-1}^{-1}(X) \simeq \Sigma^{-1} H\underline{\pi_{-1}(X)}$$

Though it is not true in general that all slices are integer suspensions of Eilenberg-MacLane spectra, we can still use this result to compute higher dimensional slices when used in conjunction with a second result concerning suspensions of the slice filtration.

We know that we cannot suspend a slice by an integral amount and necessarily get a slice in a higher dimension as in the Postinikov tower, but the slice tower does commute with suspensions by regular representation spheres. This result is again given by Hill in [**Hil12**].

USEFUL RESULT 3.2. *For any* G*-spectrum* X*, any* m*, and any* k *we have:*

$$P^{m|G|+k}(\Sigma^{m\rho_G} X) = \Sigma^{m\rho_G} P^k(X)$$

and thus,

$$P_{m|G|+k}^{m|G|+k}(\Sigma^{m\rho_G} X) = \Sigma^{m\rho_G} P_k^k(X)$$

If in the above result we replace X by $\Sigma^{-m\rho_G} X$, let $k = -1$ and apply result 3.1 we get the following:

COROLLARY 3.3. For any G-spectrum X and any m we have:

$$P_{m|G|-1}^{m|G|-1}(X) \simeq \Sigma^{m\rho_G - 1} H\underline{\pi_{-1}(\Sigma^{-m\rho_G} X)}$$

This combination of these two results provides a useful trick for computing all $(m|G|-1)$-slices of any G-spectrum X as all one really needs to compute is the (-1)-homotopy of $\Sigma^{-m\rho_G}X$. In our case, we consider $X = S^n \wedge H\underline{\mathbb{Z}}$ and $G = C_{p^k}$, and thus need only determine $\underline{\pi_{-1}(S^{n-m\rho_{p^k}} \wedge H\underline{\mathbb{Z}})}$. This can be computed rather straightforwardly as the Bredon homology $\underline{H_{-1}(S^{n-m\rho_{p^k}};\underline{\mathbb{Z}})}$ using chain complexes of Mackey functors, to obtain all (mp^k-1)-slices.

Theorem 3.4. *For all integers $n \geq 0$ we have the following slices for the C_{p^k}-spectrum $S^n \wedge H\underline{\mathbb{Z}}$:*

$$P^{mp^k-1}_{mp^k-1}(S^n \wedge H\underline{\mathbb{Z}}) = \begin{cases} \Sigma^{m\rho_G-1}H\underline{B}_{k,j} & \dfrac{n+2\cdot 0^j}{p^{k-j}} \leq m \leq \dfrac{n-2}{p^{k-j-1}},\ m,n \text{ of same parity} \\ \Sigma^{m\rho_G-1}H\underline{\mathbb{Z}}^* & n = rp^k-1 \text{ and } m = \frac{n+1}{p^k} \\ * & \text{otherwise} \end{cases}$$

While this may look a bit daunting, if we examine the spectra themselves, $\Sigma^{m\rho_G-1}H\underline{B}_{k,j}$ and $\Sigma^{m\rho_G-1}H\underline{\mathbb{Z}}^*$ are exactly the spectra given in dimensions mp^k-1 from our Key Idea 1.2. We now know of course that these are not all of the nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ but it gives a good framework upon which to build. The most obvious next question is then, "In what dimensions aside from mp^k-1 are there nontrivial slices?" At this point, another result from [**Hil12**] concerning the slice filtration is useful. In the following, H is any subgroup of G and i_H^* is the forgetful functor from G-spectra to H-spectra.

Useful Result 3.5. *The restriction to H of the slice tower of X is the slice tower of X restricted to H:*

$$P^n i_H^*(X) = i_H^* P^n(X)$$

How does this help? Let us consider an example. Suppose we wish to compute the slice tower for the C_{p^2}-spectrum $S^n \wedge H\underline{\mathbb{Z}}$. From Theorem 3.4, we know that we have nontrivial slices in dimensions mp^2-1 for various m. However, upon restricting $S^n \wedge H\underline{\mathbb{Z}}$ to C_p, we know by Result 3.5 that we should also have nontivial slices in dimensions $mp-1$ for specified m. Thus, we have determined another class of nontrivial slices that must appear in the tower. Extending this result we can easily see that for the C_{p^k} spectrum $S^n \wedge H\underline{\mathbb{Z}}$ we should have nontrivial slices in dimensions mp^d-1 for all $1 \leq d \leq k$.

It is not too difficult to see that the $mp-1$ slices determined by such computations are in fact the only nontrivial slices for $S^n \wedge H\underline{\mathbb{Z}}$ considered as a C_p-spectrum. Hence, it is not too far a leap to guess that for the C_{p^k} spectra the (mp^d-1)-slices are in fact the only nontrivial ones.

4. Colloquial summary

Just as a doctor may use a CAT scan to better understand what is happening in the human body, a mathematician often uses a given tool to better understand the structure and properties of certain mathematical objects. In addition to using the tool to study an object, one might, conversely, study the tool itself in order to better understand what it actually does and what information it gives us about the objects it is applied to.

For example, it makes sense to study what a CAT machine is and how it works because then we know what it will see in the human body, and what it will miss. The purpose of the paper [**Yar15**] is a little of both: using a given tool to study particular objects in an attempt to study the tool itself. In the paper itself, we investigate what our tool, the *slice filtration*, does to a particular family of objects called *G-spectra*. In conjunction with other work [**HHR15**] we hope that our answers might tell us more about the slice filtration itself.

4.1. The objects. The objects we consider, G-spectra, are generalizations of objects called G-spaces. As the latter are a bit more tangible, we now focus on such objects rather than G-spectra. How should one think of a G-space? First, one might think about a space as a collection of points but most often these points are all connected to form a continuous shape. For example, spaces we often consider in this context are circles, spheres, or higher-dimensional versions of spheres. Then what does the "G" tell us? G represents another mathematical object called a group. The most straightforward example of a group is a collection of numbers with an operation, like addition. A G-space is a collection of points that get jumbled around in a way that is dictated by the types of elements (or numbers) in the group G.

There is an important restriction on how the points are being jumbled: they can only be swapped around in place of other points but the overall shape of the space cannot change. For example, we might consider a circle and the way the points will be rearranged is by rotating the circle by a given amount. This type of restriction means that by using G to swap around points, we are really capturing data about the symmetry of the space.

Additionally, we can plug many different groups in for G. G could be the simplest type of group, a group containing only one number. In this case, there is actually no difference between a G-space and an ordinary space. The larger and more complicated G becomes, the more difficult it will be to study any corresponding G-space. Our goal in this setting is to study not only the shape of the object but also how the points are being rearranged.

The G-spectra that we analyze in [**Yar15**] are given the name "$S^n \wedge H\underline{\mathbb{Z}}$" where n can be any positive whole number. The part $H\underline{\mathbb{Z}}$ is an object that encodes a certain type of information. The $\underline{\mathbb{Z}}$ tells us what this type of information is being represented. The "S^n" are n-dimensional spheres and "$\wedge$" is a type of product. Essentially, the "$S^n\wedge$" part tells us to shift the object $H\underline{\mathbb{Z}}$ to dimension n. Notably, when we shift the object, we are doing it in a way that maintains its "type". It turns out that $S^n \wedge H\underline{\mathbb{Z}}$ is considered to be a relatively simple object by initial observation but our tool will allow us to see what is really happening beneath the surface, so to speak.

4.2. The tool. A common theme in many areas of study is learning about the properties of an object by seeing how the object is constructed out of smaller pieces. A microbiologist may learn about an organism by studying its cells or a chemist may study a substance by determining what elements it is comprised of.

Similarly, mathematicians will often break down objects into smaller pieces and gain insight by studying how the object is built out of the smaller pieces. The slice filtration is a tool that helps to analyze how a G-spectrum is built out of "smaller" G-spectra. This ends up being a bit difficult as it is not immediately apparent what in fact we mean by "smaller" or "simpler" G-spectra.

One way to think about the slice filtration is that it provides building blocks and a blueprint for assembling the blocks into the spectrum we are analyzing. However, the blocks it uses are a bit mysterious. If we remove the "G" and think only about spectra, we would see that a spectrum can be built as a sort of stack of uniform type blocks. Every layer is even, like building a wall out of Legos.

For a G-spectrum, our wall looks more like a game of tetris. The layers are not even; each block may have a different shape and be a part of many layers. Difficulties arise because in general we don't even know exactly what all the blocks look like! In the paper [**Yar15**] we determine the building blocks and how they are assembled to form the objects $S^n \wedge H\underline{\mathbb{Z}}$. The fact that these superficially simple objects have fairly complicated structures as determined by the slice filtration demonstrates how rich G-spectra really are.

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