Enchiridion: Mathematical User's Guides
mathusersguides.com
Volume 1 (October 2015)

A user's guide: Coassembly and the K-theory of finite groups

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1. Key insights and central organizing principles

1.1. Background. One of the main things we do as algebraic topologists is we take a gigantic object that has way too much data, such as a topological space with uncountably many points, and we distill that data down into a small, computable object like a finitely-generated abelian group. If we do our job well, the small invariant captures something essential about the big unruly space, and we can use that something to answer questions that would otherwise be out of reach.

Algebraic K-theory is a part of this larger story. It accepts as input any category C with a notion of weak equivalence and cofibration, and outputs a sequence of abelian groups $K_0(C)$, $K_1(C)$, ... In particular, we can take the K-theory of a ring, by feeding in the category of finitely-generated projective modules. Or, we can take the K-theory of a ring spectrum by feeding in the category of dualizable R-modules. It is also possible to take the K-theory of a topological ring such as $\mathbb{C}$ or $\mathbb{R}$. We can even define the K-theory of "spaces" by feeding in the category of retractive spaces over a fixed space X.

So, the algebraic K-theory machine boils the category C down to a manageable collection of abelian groups. These groups still contain essential information about C. The group $K_0(C)$ is just the free abelian group on the objects of C, but for any cofiber sequence $X \to Y \to Z$ we impose the relation $[Y] = [X] + [Z]$. So $K_0(C)$ remembers the information in C that adds over cofiber sequences, just like the Euler characteristic. The higher K-groups do not admit such a nice description, but they often contain obstructions for classical problems, such as recognizing families of finite cell complexes and constructing families of diffeomorphisms.

It's helpful to know that the groups $K_n(C)$ are actually the homotopy groups of a spectrum, or infinite loop space, $K(C)$. In particular, they form an extraordinary cohomology theory. In fact, when we take the *algebraic* K-theory spectrum of the complex numbers $\mathbb{C}$, the resulting cohomology theory $K(\mathbb{C})^0(X)$ agrees with the more familiar *topological* K-theory, $K^0(X)$. This explains why the term "K-theory" is used in both contexts.

1.2. The introduction and setup. These algebraic K-groups are very difficult to compute, so we sometimes just focus on their relationship to each other and to simpler groups. As we remark in the introduction to [**Mal15**], we get an *assembly map*

$$H_n(BG; K(R)) \xrightarrow{\alpha} K_n(R[G])$$

for any ring or ring spectrum R, and any topological group G. We seek cases where this map is injective, since that allows us to build nontrivial classes in $K_n(R[G])$ by first constructing them on the left-hand side.

Unfortunately, in homotopy theory we don't have a lot of methods for proving injectivity. The best we can do is describe α as a map of spectra

$$BG_+ \wedge K(R) \xrightarrow{\alpha} K(R[G])$$

and then produce some other map $K(R[G]) \to X$ so that the composite

$$BG_+ \wedge K(R) \xrightarrow{\alpha} K(R[G]) \longrightarrow X$$

is an equivalence of spectra. This is more than enough to conclude that our abelian-group version of the assembly map is injective.

In [**Mal15**] we investigate one such technique. Recall that $K(R[G])$ is the K-theory of the category of $R[G]$-modules that are dualizable over $R[G]$. If we instead take the $R[G]$ modules that are dualizable as R-modules, we get a different category. The K-theory of this new category may be called $G(R[G])$, the *Swan theory* of $R[G]$. It is the K-theory of *representations* of G in the category of R-modules. This functor has been carefully studied in discrete cases such as $R = \mathbb{Z}$ (e.g. [**HTW88**]), but when R is a ring spectrum such as $\mathbb{S}$, it is relatively unexplored.

Using Swan theory we are able to produce a sequence of maps

$$BG_+ \wedge K(R) \xrightarrow{\text{assembly}} K(R[G]) \xrightarrow{\text{Cartan}} G(R[G]) \xrightarrow{\text{coassembly}} F(BG_+, K(R))$$

where F stands for function spectrum. Essentially, the *coassembly* map at the end is a map from Swan theory into a kind of *cohomology* of BG. This is exciting because the two outside terms are far smaller and more computable than the two terms on the inside. In particular, one might expect that this composite has an explicit description. This is actually the main theorem of the paper:

THEOREM 1.1. *When G is a finite group, the above composite is homotopic to the equivariant norm map*

$$K(R)_{hG} \longrightarrow K(R)^{hG}$$

on the spectrum $K(R)$ with a trivial G-action.

This equivariant norm map happens to be an equivalence after certain kinds of localization, so we get a new context in which the assembly map splits. That's it for the motivation; we'll spend the rest of this section digging into the central ideas of the proof.

1.3. Key ideas of the proof. To understand where assembly and coassembly really come from, we have to re-interpret what it means to be a module over $R[G]$.

KEY IDEA 1.2. *Topological groups G correspond to connected spaces X under the identifications $X \simeq BG$ and $G \simeq \Omega X$. Modules over $R[G]$ correspond to bundles of R-modules over X. The underlying R-module is the fiber, and the G action is the monodromy.*

This idea lets us re-imagine K-theory and Swan theory of $R[G]$, for fixed R and varying G, as functors on spaces X. We use the notation $A(X;R)$ and $\forall(X;R)$ when we think this way. The assembly map then has a neat interpretation, which leads to the definition of the coassembly map (described this way in [**Wil00**] as well):

KEY IDEA 1.3. *When thinking of modules as bundles, assembly is a co-descent map, or a homotopy colimit problem map. It is a universal approximation by a homology theory in spectra. Therefore there is also a descent map, or homotopy limit problem map, or homotopy sheafification, called coassembly.*

Now we know what assembly and coassembly are. The Cartan map is just an inclusion of categories, since every dualizable $R[G]$-module is automatically dualizable over R. But how do we evaluate the composite of these things?

In Section 6 of [**Mal15**], we start from the universal properties of assembly and coassembly and produce a pair of maps that have explicit, combinatorial descriptions. The following key idea captures the result in a rough form.

KEY IDEA 1.4. *The assembly map takes each R-module M to $G_+ \wedge M$ or $\bigoplus^G M$. The coassembly map takes each $R[G]$-module N to its underlying R-module. So their composite takes each module M to a (twisted) G-fold sum of copies of M.*

This can't be a full description because we don't see what the two extra copies of BG do. In essence, they give a $G \times G^{\mathrm{op}}$-monodromy on that G-fold sum, permuting the terms of the sum using the left and right actions of G on itself. It is maybe not so obvious why this re-interpretation of assembly and coassembly is possible, but in the next section of the user's guide, we will attempt to give an intuitive explanation.

Next, we try to cut the work down as much as possible to the case of $R = \mathbb{S}$, and from there to the K-theory of finite sets, which is the sphere spectrum. This is a fairly natural plan of attack:

Key Idea 1.5. *Any natural construction on the K-theory of all ring spectra is likely to be determined by what it does to $K(\mathbb{S})$, and that in turn is often governed by what happens on $K(\text{finite sets})$.*

To make this maxim actually true, we set up our models for K-theory so that they are functors of spaces X, but still have the usual smash product pairings. So, in Section 4 of [**Mal15**], we build some Waldhausen categories of parametrized spectra. The construction of such things is generally believed to be possible, but they are rarely ever written up explicitly, perhaps out of fear that the treatment would be as long as in [**MS06**].

Finally, in Section 7 we show how to re-arrange some standard definitions of the transfer and norm maps so that they line up with the maps we saw in Idea 1.4 above. On the level of infinite loop spaces, we get a really nice and very classical picture of what a transfer is (compare with [**KP72**] and [**Ada78**]):

Key Idea 1.6. *Transfer maps are G-fold sums, where the individual terms of the sum may change their order as you move around the base. More specifically, if X is an E_∞ space, and we form a map $BG \times X \longrightarrow X$ by taking each point in X to a G-fold sum of copies of itself, with monodromy around BG that re-orders the terms of the sum, this gives a transfer map on the spectrum associated to X.*

Therefore the composite of assembly and coassembly is something that looks like a transfer. The bundle involved has base $BG \times BG$, and fiber G with $G \times G^{\text{op}}$ acting by left and right multiplication, so its total space is BG again, which is not contractible. In truth, the map we have is a transfer from $BG \times BG$ up to BG, followed by a collapse of BG to a point. By a fun geometric argument with Pontryagin-Thom collapses, we can make this line up with the equivariant norm, and that's how the theorem is proven.

References

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