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A user's guide: Coassembly and the K-theory of finite groups

Cary Malkiewich

4. Colloquial summary

I'm going to focus on the subject of topology as a whole, before zooming in on the ideas of the paper [**Mal15**].

In topology, we study shapes. You already have some idea of what a shape is: squares, circles, triangles, silhouettes of dogs, etc. are all two-dimensional shapes. There are three-dimensional shapes too, like cubes, cylinders, and helixes. It's possible to study shapes in higher dimensions as well. You might think that the study of higher-dimensional shapes is really cool and mind-blowing, or you might suspect that it's silly and pointless. But in truth, these higher-dimensional shapes are not as hard to understand as you might think, and they're also pretty important. Though we only have three dimensions of space, any mathematical model that uses 4 coordinates is actually a system that lives in abstract, four-dimensional space. If I study cancer patients, and I measure ten characteristics of each patient, those patients become data points inside a 10-dimensional space. I can work with such shapes without blowing my mind, because each point in 10-dimensional space is just a list of 10 numbers, and that's not so bad. On the other hand, it can be pretty important to understand the shape that these points form. I might learn something new about cancer by studying it closely.

Topologists have lots of fancy techniques for studying and quantifying shapes. What do we do with these tools? You might imagine that we apply them to one shape at a time, as part of a quest to "understand all shapes." But this is not quite what we do, for two reasons.

First, there really are a tremendous number of different shapes out there, just as there are many flowers in a meadow. If you've never seen the flowers in some particular area before, it can be a lot of fun to examine a few of them very closely. But carefully observing one hundred or one thousand of them would be quite a bore. Instead, you would start thinking about how the flowers in this area are different from the flowers in that other meadow, or how they're different from last year. Similarly, when we learn a new tool in topology, it's a lot of fun to try it out on a few examples. But as we dig deeply into the subject, we don't simply keep applying the same tools to more and more examples. Instead, we focus on these larger-scale patterns.

Secondly, shapes turn out to be very complicated, and even our best tools aren't powerful enough to give complete answers to the simplest questions. So even the most basic shapes, like the sphere, are not completely understood. This is both frustrating and exciting, because sometimes after a tremendous amount of work you really can understand these examples better.

We also think about much more than just spheres. We think about some really tremendous, really huge shapes. Usually we build them by taking a bunch of triangles, and tetrahedra, and so on, and we describe some recipe that says how to glue all of these pieces together. This might seem like a strange thing to do. But the shapes we build this way are much easier to study, and in some sense, every shape out there looks an awful lot like one of the shapes that we can build out of triangles.

Sometimes the shape you want to understand was created for some ulterior purpose. There is a function, the Riemann zeta function, that contains astoundingly deep information about prime numbers. This is kind of unexpected, because prime numbers are just whole numbers, you wouldn't expect them to be related to a smooth function. Similarly, there are some *shapes* that contain an unexpected amount of information about the primes. One of these shapes is called the "algebraic K-theory space of the integers." You only really need to know that this is indeed a shape, and it is quite hard to describe explicitly, but if you were able to understand all of its features, you would learn some rather difficult facts about prime numbers. The subject of *algebraic K-theory* builds lots of shapes like this. They contain really interesting information, but they are super hard to figure out.

In the paper [**Mal15**], we study something called the *K-theory assembly map*. You can think of it as a relationship between two of these shapes: one smaller, simpler shape that gets folded and deformed before it is stuffed into a larger, more mysterious shape. We are trying to understand something about this folding process. We would really like to show that, somehow, the small shape does not get destroyed beyond recognition as it is stuffed into the bigger shape. This is hard though. We don't have a small, simple picture of what's happening in the folding process. Instead, these shapes are formed by gluing together, say, thousands of triangles according to complex and arcane rules. So we can't really visualize directly what is happening to them. Instead, we have to pay attention to these gluing rules, and do some detective work to figure out how they behave as they get folded up. This is still hard though. The larger, second shape is much harder to quantify than the smaller, first shape.

However, we have a trick: we construct a third shape that will fit the second inside. Now the first shape is contained in the second, which is contained in the third, like a sequence of Russian dolls. Moreover, the first and last shapes are much easier to describe than the second. So our detective work gets a lot easier. We can show that, to some extent, that first shape was not destroyed beyond recognition, and so we understand these relationships a little bit better.

References

[Mal15] C. Malkiewich, *Coassembly and the K-theory of finite groups*, arXiv preprint arXiv:1503.06504 (2015).

Dept of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

E-mail address: cmalkiew@illinois.edu