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A user's guide: Categorical models for equivariant classifying spaces

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1. Key insights and central organizing principles

This user's guide is for the paper *Categorical models for equivariant classifying spaces*, which is joint with B. Guillou and P. May. In [**GMM**], we find models for universal equivariant bundles and their classifying spaces as classifying spaces of categories.

Equivariant bundles are, of course, a generalization of nonequivariant bundles. In this paper, we are only interested in principal (G, Π_G)-bundles $p\colon E \to B$. A principal (G, Π_G)-bundle is nonequivariantly just a principal Π-bundle, but now there are G-actions in sight everywhere, including on the structure group Π, and they need to interact compatibly with the action of the structure group Π on the total space.

Let Π and G be topological groups and suppose that we have an extension of groups

$$(1) \qquad 1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{q} G \longrightarrow 1.$$

There is a general theory of equivariant bundles corresponding to such extensions (see, for example, [**LM86, May90, May96**]). However, we will only be interested in the case when G acts on Π, the group Γ is the semi-direct product $\Pi \rtimes G$, and the extension is split.

We will refer to bundles corresponding to such extensions as (G, Π_G)-bundles: G is the equivariance group, Π is the structure group, and the subscript in Π_G denotes that G is acting on Π and the bundle corresponds to the split extension given by the semidirect product with respect to this action[1]. If the action of G

[1]In order to be consistent with [**GMM**], we do not use the notation from [**May96**] for bundles corresponding to extensions (1). Their notation is (Π, Γ)-bundles, namely the structure group

on Π is trivial, so that $\Gamma = \Pi \times G$, then we omit the subscript G, and refer to such bundles as (G, Π)-bundles[2].

Again, there is a general theory of (G, Π_G)-bundles [**tD69, Las82, LM86, May96**] corresponding to such extensions. The theory is especially familiar when G acts trivially on Π. With $\Pi = O(n)$ or $U(n)$, the trivial action case gives classical equivariant bundle theory and equivariant topological K-theory. The main result of the preexisting theory is that there is a universal principal (G, Π_G)-bundle

$$E(G, \Pi_G) \to E(G, \Pi_G)/\Pi$$

and models for the total space $E(G, \Pi_G)$ and $B(G, \Pi_G) = E(G, \Pi_G)/\Pi$ existed. However, these models were not as classifying spaces of categories.

In [**GMM**], we give models for the total space $E(G, \Pi_G)$ and the classifying space $B(G, \Pi_G)$ of (G, Π_G)-bundles as classifying spaces of categories. The reason why it is important to have such models is two-fold: they are needed in equivariant infinite loop space theory and in equivariant algebraic K-theory. We address how bundle theory comes into the picture for each of these two topics.

1.1. Motivation 1: Equivariant infinite loop space theory. Infinite loop spaces satisfy a recognition principle: they are algebras over E_∞-operads in Top (see [**May72**]). Algebras over an E_∞-operad in Cat are categories whose classifying spaces are, after group completion, infinite loop spaces. The same story carries through equivariantly for a finite group G. Equivariant infinite loop spaces (or infinite loop G-spaces) are G-spaces which have deloopings with respect to all finite dimensional representations of G, so they are zeroth spaces of genuine G-spectra. Equivariant infinite loop spaces are recognized as algebras over equivariant E_∞-operads in $G\,$Top (see [**LMS86**]).

A new development in equivariant infinite loop space theory is defining an E_∞-operad in GCat such that algebras over it are G-categories whose classifying spaces are, once group completed, infinite loop G-spaces (see [**GM**]). For this it is crucial to have models for equivariant universal bundles as classifying spaces of categories, as we go on to explain.

Nonequivariantly, an E_∞-operad $\mathscr{O}$ in Top has spaces $\mathscr{O}(j) \simeq E\Sigma_j$, namely, universal Σ_j-bundles. An E_∞-operad $\mathscr{O}$ in Cat is defined by the property that the space-level operad $B\mathscr{O}$ with spaces $B\mathscr{O}(j)$ is an E_∞-operad in Top. Let $\widetilde{\Sigma}_j$ be the category with objects the elements of Σ_j and a unique morphism between any two objects. Therefore any object is both initial and terminal, and $\widetilde{\Sigma}_j$ is a contractible category. Also, it has a free Σ_j-action, so $B\widetilde{\Sigma}_j \simeq E\Sigma_j$, thus

is listed first and the extension group second. In the notation from [**May96**], the bundles we are considering are $(\Pi, \Pi \rtimes G)$-bundles.

[2]The notation for bundles corresponding to extensions with $\Gamma = \Pi \times G$ is consistent with [**May96**], where they adopt the same convention for the trivial action case, and we felt that our notation for the general case better generalizes this.

the categorical operad $\mathscr{O}$ with categories $\mathscr{O}(j) = \widetilde{\Sigma}_j$ is an E_∞-operad. This is also known as the *Barratt-Eccles operad*, and algebras over $\mathscr{O}$ are permutative categories [**May74**].

The definition of an equivariant E_∞-operad $\mathscr{O}_G$ in $G\,\mathrm{Top}$ is in terms of equivariant universal bundles: the spaces $\mathscr{O}_G(j)$ are defined to be universal (G, Σ_j)-bundles, which we denote for now as $E(G, \Sigma_j)$. These are universal principal Σ_j-bundles, with total and base G-spaces, G-equivariant projection map, and commuting actions of G and Σ_j on the total space. Models for universal equivariant bundles and their classifying spaces are described in [**May96**, VII], for example, but they are not given in terms of classifying spaces of categories.

An E_∞-operad $\mathscr{O}_G$ in $G\mathrm{Cat}$ is defined by the property that applying the classifying space functor levelwise yields an E_∞-operad in $G\,\mathrm{Top}$. Thus finding an E_∞-operad in $G\mathrm{Cat}$ amounts to finding models for equivariant universal principal (G, Σ_j)-bundles as classifying spaces of G-categories. We summarize this in Table 1 below.

	A nonequivariant E_∞- operad $\mathscr{O}$	An equivariant E_∞- operad $\mathscr{O}_G$
in Top	has spaces universal Σ_j-bundles, i.e., $\mathscr{O}(j) \simeq E\Sigma_j$ **example:** $\mathscr{O}(j) = B\widetilde{\Sigma}_j$	has spaces universal (G, Σ_j)-bundles, i.e., $\mathscr{O}_G(j) \simeq E(G, \Sigma_j)$
in Cat	is defined such that $B\mathscr{O}(j) \simeq E\Sigma_j$ **example:** $\mathscr{O}(j) = \widetilde{\Sigma}_j$	is defined such that $B\mathscr{O}_G(j) \simeq E(G, \Sigma_j)$

Table 1: E_∞ operads

From the table, we can see that in order to have a definition of $\mathscr{O}_G$ in Cat, for each j, we need a category $\mathscr{O}_G(j)$ whose classifying space is a universal principal bundle $E(G, \Sigma_j)$.

1.2. Motivation 2: Equivariant algebraic K-theory. Quillen's first definition of higher algebraic K-groups was as the homotopy groups of a space $BGL(R)^+$, which turns out to be homotopy equivalent to the basepoint component of the group completion of the topological monoid $B(\coprod_n GL_n(R)) = \coprod_n BGL_n(R)$. Note that this is the topological monoid of classifying spaces of principal $GL_n(R)$-bunldes under Whitney sum. Equivariantly, we are unconcerned with any variant of Quillen's original plus construction, but we instead replace the classifying spaces of principal $GL_n(R)$-bundles by classifying spaces of equivariant principal bundles, before group completion.

Note that in contrast to the equivariant bundles considered in the previous section, when G was not acting on Σ_j and we had commuting actions on the total space, now we are assuming that G acts on R, which induces an action on $GL(R)$. The whole point is to take this action into account. The bundles which we are trying to understand are $(G, GL_n(R)_G)$-bundles; they are universal principal $GL_n(R)$ bundles, but they have twisted actions on the total space, i.e., they have an action of the semidirect product $GL_n(R) \rtimes G$ on the total space. The base space is a G-space and the projection map is G-equivariant.

The intuition of defining the equivariant algebraic K-theory space of a G-ring in terms of classifying spaces of $(G, GL_n(R)_G)$-bundles is right, in the sense that we are rigging the spaces to provide an algebra over an E_∞-operad in $G\,\mathrm{Top}$ that can be fed into an equivariant infinite loop space machine. We refrain to say more about this here, because algebraic K-theory is not really the topic of [**GMM**]; however, the motivation for me was to use these results in my thesis work on equivariant algebraic K-theory. The main result of [**GMM**] is in a sense the starting point of my thesis.

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