Enchiridion: Mathematical User's Guides
mathusersguides.com
Volume 1 (October 2015)

A user's guide: Categorical models for equivariant classifying spaces

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3. Story of the development

The paper [**GMM**] was the starting point of my thesis work. In Section 1 of this user's guide, I gave two motivations for the main result of the paper: one was that it motivates the definition of equivariant algebraic K-theory, which is what I was beginning to work on, and the second was that it provides a model for an E_∞-operad in G-categories, which in turn lets one develop operadic equivariant infinite loop space theory, and this was what my co-authors Bertrand Guillou and Peter May were working on. This was a fortunate collision of interests which resulted in jointly writing this paper, and a lot of subsequent joint work that naturally followed.

By the end of my second year of graduate school, my interest was totally piqued by algebraic K-theory because of its connections to numbers theory, which was what I initially wanted to study before being lured by Peter May into the world of algebraic topology. I spent a long time learning about algebraic K-theory for my topic exam, and I was thirsty for more. On the other hand, Peter May enticed me with the wonderful world of equivariant homotopy theory, and that is how the general idea of equivariant algebraic K-theory ossified into a thesis project.

Now "equivariant algebraic K-theory" had to be made sense out of, and the first problem my adviser, Peter May, suggested was to prove an "equivariant plus=Q" theorem. The classical "plus=Q" theorem is a deep result of Quillen stating that two very different definitions that he gave of algebraic K-theory of a ring agree, namely his "plus construction" and his more general "Q-construction" for exact categories, when specified to the category of finitely generated projective modules over the ring.

I had no idea at that point what an equivariant version of this would be. Peter directed me to the relevant literature, which consisted of a paper by him, Fiedorowicz and Haushild from the 1980's in which they study a space-level version of equivariant algebraic K-theory for rings with trivial G-action [**FHM82**] and some old papers by Dress and Kuku in which they give and study a definition of equivariant algebraic K-groups for exact categories with trivial G-action [**DK82**]. I formulated a clear task for myself, which was to reconcile these definitions. I was really thrilled when I worked this out, because it was the first problem I had gotten. When I excitedly told Peter I had figured it out, he was very pleased but he told me that the work has just begun. And so it was.

The definitions I had reconciled only worked for trivial G-action on the input ring. My next goal was to generalize these definitions to rings with nontrivial action. The equivariant "plus" construction definition from [**FHM82**] is in terms of classifying spaces of equivariant $GL_n(R)$-bundles, but the kind where the equivariance group G does not act on the fiber $GL(R)$. Even in that setting the model used for these classifying spaces was very unwieldy. One of the main ingredients in my reconciliation of their definition with Dress and Kuku's definition was to use a different model for the classifying space of a $(G, \Pi \times G)$-bundle, one that was the classifying space of a category, and which I had learned from a working draft of Bertrand Guillou and Peter May on equivariant infinite loop space theory. Thus my main task narrowed down to bumping up their model to the case of twisted actions – the different complexity levels of the equivariance were described in the previous section.

After working on this for a while during the summer of 2011, I showed that the categorical model for the total space of equivariant universal principal bundles from Bert and Peter's note generalizes to the case when G acts on the fiber group. The main idea was to replace group homomorphisms that were showing up with crossed homomorphisms. However, going into this project I did not know what a crossed homomorphism was, and the breakthrough definitely happened when I came across the definition of a crossed homomorphism and I realized that a lot of things I was encountering were crossed homomorphisms, which were central to understanding the twisted actions.

The classifying space for an equivariant principal bundle is the quotient of the total space by the structure (fiber) group. In order to obtain the desired classifying space that I was actually using for comparing the "plus" and "Q" definitions even in the case of trivial G-action, one has to show that the quotient of the total space with the structure group satisfies two commutations that were not obvious. However, neither Peter and Bert in their note, nor I had written a proof of the commutations that would give the right classifying space, and without them my comparison result of "plus=Q" in the case of trivial G-action and its generalization to nontrivial G-action would fall apart.

That was the point when Peter suggested all three of us jointly work to finish off this project on categorical models of classifying spaces of equivariant bundles and make it a separate paper since the results are of independent interest. The work was laid out for us at this point, and we carried out the verifications that we needed. Then we also computed the fixed point spaces of the classifying spaces of these equivariant bundles in terms of fixed point categories, and a very pleasant surprise arose when during discussions with two number theory postdocs at UChicago at the time, Matthew Morrow and Liang Xiao, I have learned about nonabelian cohomology, and realized that first nonabelian cohomology sets $H^1(G, GL_n(E))$ appeared as the isomorphism classes of objects of these fixed point categories when G is the Galois group of a finite field extension E/F. This was crucial when I later on had to use the result from our paper in my thesis on equivariant algebraic K-theory.

This paper has been in many ways a starting point. It was not only the starting point of my work on equivariant algebraic K-theory, but it was also the beginning of a long collaboration with my adviser Peter May and Bertrand Guillou that is still ongoing, and that Angelica Osorno who was then a postdoc at UChicago also became a part of. I have been very lucky to learn from all of them over the years.

References

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