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A user's guide: Categorical models for equivariant classifying spaces

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4. Colloquial summary

In topology we study spaces, which are sets of points with a notion of nearness. Some examples of spaces are a point, a finite number of points in the plane, a circle, a sphere, a higher dimensional sphere, Euclidean space, a doughnut shape - these are all easy to visualize. From spaces we know, we can create new spaces: For example, if we take two circles and take their "product," which is the space that we obtain by sliding one circle around the other one, we obtain the doughnut shape. Another construction that gives this same space is taking a rectangle, glueing the top edge to the bottom edge, and glueing the left edge to the right edge. Just using basic constructions like these, we can get bigger and bigger spaces. Most spaces that a topologist considers are infinite-dimensional and very complicated, but they are built out of basic building blocks called cells, which are glued together according to a given recipe.

In the eyes of a topologist two spaces are deemed equivalent if one can continuously be transformed into the other without breaking it apart. For instance, a circle and a square are equivalent because one can be continuously deformed into the other. However, a sphere is, for example, not equivalent to a circle, because there is no way to deform it into a circle without breaking a hole, or rather two holes in it. The Euclidean plane is equivalent to a disk centered around the origin, you can imagine shrinking the plane from all directions into a disk. Now you can further shrink the disk into its midpoint. So the plane is equivalent to the disk, which in turn is equivalent to a point. Through the lens of algebraic topology these spaces are the same. But clearly the entire Euclidean plane is infinite and much more unwieldy than the disk. The disk still has infinitely many points, so it is still much more complicated than just the point. So from all these models for the same topological space up to equivalence, the point is clearly the easiest one to work with.

There are some spaces in algebraic topology that are only defined up to equivalence: we know that there exists a space satisfying a property X and all spaces satisfying property X are equivalent. An example of a space defined up to equivalence is the "classifying space of a bundle with certain fiber." A bundle consists of two spaces, a base space and a total space, and a projection map from the total space to the base space such that the preimage of each point in the base under this projection map, called the fiber of the bundle, has the some nice structure. What it means for a space to be a classifying space for a bundle with a certain fiber is that every other bundle with that same fiber corresponds in a precise way to a map from its base space to the classifying space. Given a bundle with a certain fiber, its classifying space is defined up to equivalence: we know there exists a space with the property we just described and also that any other space that has the described classifying property is equivalent to it. Therefore, when it comes to classifying spaces of bundles, one needs to pick and work with one of many possible models for the same space.

In the paper [**GMM**] we consider "equivariant bundles," which are bundles that have additional structure on them. Very roughly, they are bundles with the additional data of their symmetries. The classifying spaces for equivariant bundles also have extra data that encodes symmetries. There were previously known models for such classifying spaces, but they were huge and unwieldy spaces. As we have seen above, there can be many models of the same space, which are all equivalent, some easier to understand than others. In the paper we find models for classifying spaces of equivariant bundles that are more convenient to analyze for certain applications than the previously known models. The equivalent models that we find arise from categories - a category is a collection of objects and directed relations between them, called morphisms, or simply maps. The spaces built out of categories are infinite dimensional, but they follow a very precise recipe dictated by the objects and relations between them, and can therefore be easier to understand than other spaces. Of course, it all depends on the complexity of the category in question, but the models that we find come from uncomplicated categories with very simple relations between the objects.

References

[GMM] B. J. Guillou, J. P. May, and M. Merling, *Categorical models for equivariant classifying spaces*, arXiv:1201.5178v2.

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