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A user's guide: Bousfield lattices of non-Noetherian rings

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1. Key insights and central organizing principles

Conceptually, the starting point for [**Wol14**] is the paper [**DP08**] by Dwyer and Palmieri, studying the Bousfield lattice of the derived category of a certain non-Noetherian ring Λ_k, where k is a countable field.

As explained in the Introduction in [**Wol14**], and in the Philosophy section in [**DP08**], there are formal parallels between this lattice, and the Bousfield lattice of the p-local stable homotopy category $\mathcal{S}$, which arises in topology. The latter is very bizarre and hard to grasp, and derived categories of non-Noetherian rings offer a somewhat simpler setting. In the analogy between topology and algebra, it becomes natural to consider Dwyer and Palmieri's ring Λ_k, but with the p-local integers $\mathbb{Z}_{(p)}$ replacing k. This is $\Lambda_{\mathbb{Z}_{(p)}}$.

Then the obvious, and somewhat uninspired, research question is: What results of [**DP08**] carry over to $D(\Lambda_{\mathbb{Z}_{(p)}})$?

The results in the paper [**Wol14**] build towards a fairly decent answer to this question. Various tools and machinery are developed along the way. For each such tool, it is natural to try to state and prove results in as much generality as possible. Thus the paper progresses from general to specific, in three distinct phases, Sections 3 – 5. This is the literal order of exposition, but also a helpful conceptual roadmap. Each phase has a very different conceptual flavor.

1.1. Intro and Section 2. Before discussing Sections 3 – 5, it's worth saying some things about the first two sections. The Introduction and Section 2 were intended to be a careful, somewhat thorough introduction to Bousfield lattices: what they are, why they are interesting, and examples of what we know about them. This stemmed from two insights.

Key Idea 1.1. [**IK13**, Thm. 3.1] *In any well generated tensor triangulated category, there is a set of Bousfield classes (rather than a proper class), and thus a Bousfield lattice.*

Key Idea 1.2. *The proof of* [**IK13**, Thm. 3.1] *also applies to well generated localizing tensor ideals of a tensor triangulated category.*

The first idea suggests that Bousfield lattices show up in more places than previously realized, and so deserve a unified treatment. But the second idea is more exciting: if $\mathsf{S} \subseteq \mathsf{T}$ is such a localizing tensor ideal, then we can talk about its Bousfield lattice $\mathsf{BL}(\mathsf{S})$. But now the tensor unit may not be in S! This causes many previous known facts about Bousfield lattices to break. For example, it is no longer the case that $\langle X \rangle = \langle 0 \rangle$ implies $X = 0$, the maximum Bousfield class is no longer $\langle \mathbb{1} \rangle$, and complements are no longer unique.

It becomes necessary to carefully revisit what we know about Bousfield classes, and treat them in this more general light. This is what Section 2 is about. I also included a long subsection on examples, to gather facts that I thought were illuminating in getting a sense of what Bousfield lattices can be like.

1.2. Section 3. The key idea in Section 3 is the following.

Key Idea 1.3. *The Verdier quotient T/S of a well generated tensor triangulated category T is also well generated. So it has a Bousfield lattice. What is it? There is a quotient functor $\pi : \mathsf{T} \to \mathsf{T}/\mathsf{S}$; how does this connect the two Bousfield lattices?*

Sure enough, the quotient functor induces a map between the Bousfield lattices. You might expect this is trivial, but in fact I think it's only true if $\mathsf{S} = \langle Z \rangle$ is a Bousfield class! In other words, you can't just mod out by anything, it has to be a Bousfield class. I took that to mean that at least something non-trivial is going on.

Section 3 runs with this idea. Lemma 3.1 gives an onto join-morphism. When you have an onto map, you try to take the kernel and sharpen it. Thus it's natural to consider the quotients of $\mathsf{BL}(\mathsf{T})$. And then you have the following clever question.

QUESTION 1.4. *What is the relationship between the Bousfield lattice of a quotient, and the quotient of a Bousfield lattice?*

Most of the results in Section 3 center on this question. Proposition 3.2 seems to give a good answer, but really points out a key weirdness about lattices. The map in Proposition 3.2 is onto, has trivial kernel, and preserves arbitrary joins. But this isn't enough for a lattice isomorphism! Proposition 3.5 gives two examples where this is *not* a lattice isomorphism. On the other hand, in Corollaries 3.3 and 3.4, we have two cases where there *is* an isomorphism. The more important one is Corollary 3.3. This is a more-or-less satisfying answer to Question 1.4.

1.3. Section 4. Section 4 is centered on trying to understand how a map of rings $f : R \to S$ can relate the Bousfield lattices $\mathsf{BL}(D(R))$ and $\mathsf{BL}(D(S))$. Some results are in complete generality, true for any ring map f. But the more interesting ones are in Section 4.3, when we add a hypothesis.

The reader is quickly confronted with a severe warning, and it's worth pointing out that this is *not* a key insight or central idea. The issue is simply this: the rings Λ_k and $\Lambda_{\mathbb{Z}_{(p)}}$ are graded, and then their derived categories are bigraded, and this is a little unusual. I wanted my statements and results in this section to hold for them. But I also wanted to talk about the rings that more people cared about, i.e. ungraded rings. Since all the proofs applied in both cases, the most streamlined way I could think to explain things was by laying out the warnings.

KEY IDEA 1.5. *A ring map $R \xrightarrow{f} S$ induces maps $f_* : \mathsf{Mod}\text{-}R \rightleftarrows \mathsf{Mod}\text{-}S : f^*$, which descend to derived categories. These are very, very nice functors. For example, the projection formula holds.*

Section 4.1 is full of helpful properties of these functors. The projection formula is probably the single most useful result in the whole paper, and gets used over and over in proofs. Projection formulas like this appear in many other contexts; this setup is sometimes called the "Grothendieck context".

The results in Section 4.2 are mostly partial and unsatisfying, only precursors to their improvements in Section 4.3. The condition there – $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X – is a strong and useful one. It's essentially a surjectivity condition, and so I expect it to hold in situations where R is "big" and S is "small". (For example, in my thesis I show that this condition is satisfied when $f : R \to S$ is a surjection onto a Noetherian ring.)

It is also a somewhat arbitrary condition. A more complete treatment would have considered other types of ring maps (injective?), or other conditions. But in the paper there's only this one condition considered; it is the one that yields nice results (of these I think Theorem 4.18 is the nicest), and applies in Section 5.

1.4. Section 5. Finally in Section 5 we have specialized to the original question: what can we say about the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$?

KEY IDEA 1.6. *The ring $\Lambda_{\mathbb{Z}_{(p)}}$ is closely related to the two rings $\Lambda_{\mathbb{F}_p}$ and $\Lambda_{\mathbb{Q}}$, which are covered in the Dwyer-Palmieri paper* [**DP08**] *because $\mathbb{F}_p$ and $\mathbb{Q}$ are countable fields.*

For example, we have the projection $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$. Since it's surjective, we can ask if the condition of Section 4.3 holds. The first order of business is Proposition 5.3, showing that it does. For $g_\bullet$ at least; my guess is that $h_\bullet$ does not satisfy the condition of Section 4.3, but I was unable to prove or disprove it. Thus we can apply all the results from Section 4, using $g_\bullet$. This gives various lattice maps, and lattice isomorphisms with quotients. But the strongest results come from two realizations.

KEY IDEA 1.7. *The three derived categories $D(\Lambda_{\mathbb{Z}_{(p)}})$, $D(\Lambda_{\mathbb{F}_p})$, and $D(\Lambda_{\mathbb{Q}})$ are related by a smashing localization functor $L : D(\Lambda_{\mathbb{Z}_{(p)}}) \to D(\Lambda_{\mathbb{Z}_{(p)}})$, finite localization away from $\mathsf{th}(g^\bullet \Lambda_{\mathbb{F}_p})$.*

KEY IDEA 1.8. *Any smashing localization causes a splitting of the Bousfield lattice into a product of lattices.*

From the first realization we can perform many computations, and Proposition 5.8 gives a clear picture of what's equal to what. But the second realization, laid out in Lemmas 5.11 and 5.12, Proposition 5.13, and Theorem 5.14, is more structurally powerful. Combining the two, we get a very explicit and satisfying splitting of the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$.

At the end, Corollary 5.18 is a nice non-trivial application of the theory developed in Section 2, on Bousfield lattices of localizing tensor ideals that don't contain the tensor unit. Corollary 5.19 is only included for the record; cardinality calculations are useful qualitative facts to have available.

2. Metaphors and imagery

The central objects of the paper are Bousfield lattices of derived categories of non-Noetherian rings, and this triple-layer idea has three levels of imagery.

2.1. Bousfield lattice. Bousfield lattices are quite visual and explicit in my mind. They feel simpler – simpler than the categories they're built out of, at least. Considering "the Bousfield lattice of..." involves a mental stepping-above, moving up a level. This is usually experienced as moving up in my visual field, or less commonly lifting up off the page. Each symbol $\mathsf{BL}(-)$ evokes this cleaner, more transparent feeling. Bousfield lattices are posets, with a bottom and a top. I see vertices and lines connecting them, indicating the partial order, moving from bottom to top. The join of a set of elements is obviously positioned above them somewhere, the meet below. These images are obvious and were probably learned in a classroom or from a textbook. With Bousfield lattices, the joins – even infinite joins – are easy to compute, and show up frequently.

The smash operation also appears below its arguments, further down than the meet. It feels stable and comforting. For the smash commutes with coproducts, hence arbitrary joins. It's not too hard to compute this operation, and manipulate it efficiently. The meet, on the other hand, is hard to pin down, is ideally avoided in computations, and exists in a visual position but a vague one.

While computing with Bousfield classes, as in Section 2 of the paper, there's a fair amount of symbol manipulation. This is very formal; I know various manipulations and formulas, and can apply them when appropriate. They wait behind me, out of my vision, but come forward when something on the paper starts to ring a bell. It's almost a form of pattern recognition; at each step the calculations displayed before me are suggesting operations to perform. I can only go forward about two steps in my head, without writing things down.

If prompted to contemplate the structure of the Bousfield lattice, I immediately see the statement $\mathsf{BA} \subseteq \mathsf{DL} \subseteq \mathsf{BL}$. I know that in many cases they are all equal, but in the cases I'm mostly fascinated by they aren't. There's a big mysterious gap between BL and DL, and floating there are the square-zero objects, represented by the notorious $I \wedge I = 0$. Both DL and BA have the feel of nicer, more structured lattices, albeit sitting inside BL awkwardly. The objects in BA appear to me in pairs, since these are the complemented classes. There is a special sub-collection among them, the pairs $\{\langle C\mathbb{1}\rangle, \langle L\mathbb{1}\rangle\}$ arising from smashing localization functors.

Once on a hike in the rain, I looked at the drops falling in a puddle on the trail, and I thought I saw something complex enough, rich enough, to be "what a Bousfield lattice looks like". Not in a way that "I understand BL now", but rather I understood it could be understood, even if it were so complex that that understanding wouldn't take a form we had yet conceived of. Although sadly I won't be able to describe the image, I can sort of see it in my mind's eye, and I evoke this puddle experience (because it was as much about the image as the rain, the scenery, and the hotspring I was leaving) when I need to remember what I'm dealing with and why.

2.2. Derived categories. The key to understanding and being able to work with derived categories is establishing a strong hierarchy of conceptual levels. Here I'll just talk about derived categories of rings. A ring R gives a module category $\mathsf{Mod}\text{-}R$. This needs to be treated as a black box, a nice abelian category where almost everything works as we would want it to. The category of chain complexes $Ch(R)$ has a flavor completely of its own – its own obvious visual images of complexes of modules, its own body of results and computations. Every time I inhabit this category, I see commutative diagrams between chain complexes, and computations extracting information from these.

But to pass to $D(R)$, I strongly suggest reorienting. Forget chain complexes as much as possible. This category $D(R)$ has a host of formal properties, and it is most helpful to clean the plate and start from scratch with these. The approach of axiomatic stable homotopy theory obviously encourages this. Over time, a map of objects $X \to Y$ becomes exactly that – just two symbols connected by an arrow. No chain complex representatives, no modules, no rings acting. I know there's a tensor operation – heuristically a multiplication – with a unit. There are coproducts, which I think of as an addition operation. The triangles are fun. They take some getting used to, but after enough practice seeing how one uses them in computations, they become powerful and enjoyable tools. Every morphism fits into such a triangle. Morphisms between triangles behave nicely. The ability to turn a triangle into a long exact sequence of homology groups imbues the triangles with a strength and sharpness.

The way to think about derived categories is to work with these formal properties as much as possible. An object is just an object. But it inevitably becomes necessary to dig deeper, into the structure of the particular axiomatic stable homotopy category you are working with. And so you must choose a representative, write down a chain complex, and stare at it. Knowing when to pass from the $D(R)$ level to the $Ch(R)$ level, and back up, is an art.

I've found it helpful to nurture the distinct conceptual flavors, and atmospheres, of the two levels $D(R)$ and $Ch(R)$. Each has developed its own ecosystem in my mind. This helps me to immerse myself in the subtleties and functionalities of one level, with less distraction and confusion. Moving between levels is a discrete cognitive act, and then I'm in a new environment, with its own flavor. The ability to nurture this separation of ideas, and navigate between, seems crucial to understanding abstract mathematics and harnessing its power.

To give an example, consider Definition 5.6. The inclusion $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$ of the p-local integers into the rationals evokes basic abstract algebra imagery: these rings are sets of elements with structure; ring maps are somewhat rigid; we can imagine writing out fractions a/b. Then I contemplate that this induces a morphism $\Lambda_{\mathbb{Z}_{(p)}} \to h^\bullet \Lambda_{\mathbb{Q}}$ in $D(\Lambda_{\mathbb{Z}_{(p)}})$. First I see the inclusion passing to an inclusion $\Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$ (see comments below). Then I imagine these as chain complexes, concentrated in degree zero, in $Ch(\Lambda_{\mathbb{Z}_{(p)}})$. You can see this clearly: two infinite complexes, mostly zeros, with an arrow mapping from $\Lambda_{\mathbb{Z}_{(p)}}$ to $\Lambda_{\mathbb{Q}}$. This arrow, a $\mathbb{Z}_{(p)}$-module map, still prompts me to think about the elements inside these modules. But then we switch to the derived category level. I see the target complex as a representative of $h^\bullet \Lambda_{\mathbb{Q}}$, and suddenly there is only two symbols connected by an arrow: $\Lambda_{\mathbb{Z}_{(p)}} \to h^\bullet \Lambda_{\mathbb{Q}}$. The symbol $h^\bullet$ reminds me this is a high-level functor, between derived categories that are themselves now zoomed-out black boxes. Now, in this triangulated world, it is a simple step to extend this map to a triangle, and define F as the fiber. At this point in my mind, F really exists just as a symbol completing a triangle.

2.3. Non-Noetherian rings. The paper is very notation-heavy, and I think there's a real danger of getting lost by the sub-subscripts. At no point do I use the substructure of, say $\Lambda_{\mathbb{Z}_{(p)}}$, to prove anything; there are no proofs manipulating the x_i's or n_i's. For this reason, I think it's helpful to mostly forget about what $\Lambda_{\mathbb{Z}_{(p)}}$ is, except for being a non-Noetherian polynomial $\mathbb{Z}_{(p)}$-algebra. Then $\Lambda_{\mathbb{Q}}$ is the same, but with a $\mathbb{Q}$ everywhere that $\Lambda_{\mathbb{Z}_{(p)}}$ has a $\mathbb{Z}_{(p)}$, and likewise $\Lambda_{\mathbb{F}_p}$. Most of my intuition about how $\mathbb{Z}_{(p)}$, $\mathbb{F}_p$, and $\mathbb{Q}$ relate gets carried over to $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$. This certainly helps in suggesting to me what might be true about the latter; proving these statements requires care, of course.

I never really use the non-Noetherian nature of these rings, except insofar as I build on results in [**DP08**]. (In that paper, they certainly compute with x_i's.) The key point is that Noetherian rings are too nice. The Bousfield lattice of the derived category of a Noetherian ring is completely understood, and is too nice (in my opinion). These rings $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$ are some of the simplest non-Noetherian rings to consider.

2.4. Other concepts. Next I'll describe how I think about a few of the other ideas that show up in the paper.

2.4.1. *Tensor triangulated categories.* I discussed this above, when talking about the formal properties of $D(R)$ and working at that level. Since this formal structure is all that tensor triangulated categories have, by definition, one has no choice but to use that and only that. But even when proving things about tensor triangulated categories, I find it useful to occasionally "project down" to one of the colorful examples, like spectra or $D(R)$. This might suggest intuition about what to expect to be true. But it also nicely enriches the dry category-theoretical reasoning, with extra flavor and (unnecessary but still attractive) details.

Well generated categories are hard to grasp. Their definition is complicated, and it takes a long time to build intuition. I find it best to treat them as "the right class of triangulated categories". That is, they are closed under taking subcategories and quotients, more or less. The more specialized class of compactly generated triangulated categories is not. Most results about well generated categories that I feel I understand, are natural generalizations of results on compactly generated categories. So the essence is the same: with arbitrary coproducts, the full category is "large"; we ask for a smaller generating set, which helps us do "small" things; in good situations results about small things can be generalized to the whole large category. For compactly generated categories, the small things are quite small; in well generated categories we allow for less-small things. Thus a compactly generated category is the same as an $\aleph_0$-well generated category, but going less-small we consider regular cardinals larger than $\aleph_0$.

2.4.2. *Lattice quotients and products.* I don't feel that I have good metaphors and imagery for these concepts, and this prevents me from feeling like I understand what's going on in some of my results about them.

I certainly don't have much intuition about a lattice quotient, although it is reassuring to remember that $\mathsf{L}/a{\downarrow} \cong a{\uparrow}$ (see Section 2.3 of [**Wol14**] for definitions). The sublattice $a{\uparrow}$ invokes a strong, and obvious, visual image. Otherwise, I treat any object $\mathsf{BL}(\mathsf{T})/J$ quite formally, going back to its definition. Similarly, I am befuddled about the Bousfield lattice splitting induced by smashing localizations, described in Theorem 5.14. I feel like I should be able to "see" what such a product looks like, but haven't quite gotten there. Results like Proposition 5.13 certainly help.

2.4.3. *Bousfield lattices of subcategories.* Lattices of the form $\mathsf{BL}(\mathsf{loc}(X))$, where $\mathsf{loc}(X) \subseteq \mathsf{T}$, seem mysterious, in a good way. Section 2 facts, like that complements aren't unique, and the maximum is $\langle X \rangle$, are intriguing. I'm comfortable working in this Bousfield lattice, slowly, but find it confusing to tease out the relationship between this lattice and the full Bousfield lattice $\mathsf{BL}(\mathsf{T})$. Definition 5.10 and Lemma 5.12 were inspired by my attempt to clarify this. Fortunately, in the case of a smashing localization, as in Section 5, these are well-behaved.

2.4.4. *Localization.* Localization functors don't feature too prominently in this paper, but when they show up, in Section 5, they are very powerful tools. It took me a long time, and lots of practice, to get comfortable with localizations. Maybe all that I will say here is, I've found it useful to alternate between the localization perspective and the Verdier quotient perspective.

Every localization on T is determined by its acyclics, a nice reliable localizing subcategory S. The existence of such a localization is equivalent to the Verdier quotient T/S having Hom sets, i.e. being a triangulated category on its own. This category T/S is easy to think about, formally at least: it is a "projection" of T where everything in S (and maps that factor through an object in S) are killed. (For years I fought the murderous imagery, but reluctantly am adopting it; there's no concise alternative.) The quotient map $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ has a right adjoint, and the localization on T is just the composition of these.

One key idea is this: that right adjoint induces an equivalence between T/S and the image of the localization functor, i.e. the category of locals. This equivalence gives the category of locals its own tensor triangulated structure. So when thinking about local objects, I alternate between considering quotient objects in T/S, and considering honest objects of T but with their own tensor product and coproducts. In the case of a smashing localization – a very, very nice type of localization – these are the same as in T.

2.4.5. *The functors $f_\bullet$ and $f^\bullet$.* The adjoint pair $f_\bullet : D(R) \rightleftarrows D(S) : f^\bullet$ is the central idea of Section 4, and has a lot of associated imagery and metaphors. Both $f_\bullet$ and $f^\bullet$ feel like good, well-behaved, nice functors, that have good properties that one can get a handle on. Computing with them is straightforward. There is a concrete image of $D(R)$ on the left, $D(S)$ on the right, and the two functors moving objects and morphisms back and forth.

The left adjoint $f_\bullet$, which sends R to S and preserves compact objects, seems to be the nicer of the two, a more handy machine. It's more structural, like an architect. If it were colored, it would be green or blue. The right adjoint $f^\bullet$, which is just a forgetful functor, acts by taking an object in $D(S)$ and simply plopping it down into $D(R)$. The consequence is more delicate, and has more personality. I would say $f^\bullet$ is peaceful and quiet, maybe off-white, a light yellow, or gentle gray.

The fact that both $f_\bullet$ and $f^\bullet$ give order-preserving maps on Bousfield lattices has a definite visual image. I see $\langle X\rangle \leq \langle Y\rangle$, and then I see $f_\bullet\langle X\rangle \leq f_\bullet\langle Y\rangle$ transported to the right, in the same spatial orientation but maybe stretched or translated.

The projection formula was mostly absorbed, over time, as one of the various options for manipulation of formulas, as discussed above when discussing Bousfield lattices. It is so important and useful, though, that it's one of the first patterns I try to apply and take advantage of. During the most focused time of working on this paper, I think the projection formula, and the Corollary 4.5 version of it, became so saturated in my thoughts that I often performed Bousfield class computations in parallel in my mind – one version in $\mathsf{BL}(R)$ and one version in $\mathsf{BL}(S)$. The projection formula is a way of moving back and forth, and I reached the point where I maintained both lines of reasoning simultaneously.

As mentioned previously, the condition in Section 4.3 – that $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X – strikes me as a surjectivity condition, heuristically. In this context, I visualize $\mathsf{BL}(R)$ larger than $\mathsf{BL}(S)$, and mapping onto it, with $\mathsf{DL}(R)$ and $\mathsf{BA}(R)$ mapping onto their counterparts. The functor $f^\bullet$ embeds $\mathsf{BL}(S)$ (and $\mathsf{DL}(S)$ and $\mathsf{BA}(S)$) into $\mathsf{BL}(R)$ in an intriguing way. For example, the statement $\langle f^\bullet S\rangle \vee \langle M_f\rangle = \langle R\rangle$ is just stating that $\langle S\rangle$ in $\mathsf{BL}(S)$ (which is complemented there by $\langle 0\rangle$) gets sent to a complemented class in $\mathsf{BL}(R)$ (in which case its complement is now $\langle M_f\rangle$).

3. Story of the development

I have broken the story of development into two parts: logistical and conceptual.

3.1. Where and when the ideas arose.

This section sets the stage for the next, by describing the context of where and when the ideas arose. The details are logistical and psychological.

The idea to investigate $D(\Lambda_{\mathbb{Z}_{(p)}})$ came during the summer of 2009, while I was riding the Trans-Siberian railway between the east coast of Russia and Irkutsk. It was the summer after my third year of math graduate school; my newly established PhD advisor suggested I read [**Mar83**] and [**HPS97**] to become familiar with this axiomatic approach to tensor triangulated categories and stable homotopy categories. I had recently read [**HS98**], [**HP99**] and [**DP08**], so my mind was primed for drawing a stronger connection between the p-local stable homotopy category $\mathcal{S}$ and the derived category $D(\Lambda_k)$. But it wasn't that exciting of a research question, so I didn't pursue it very far.

In the spring of 2011 I took a leave of absence from grad school, and went to Thailand. My base was Tonsai – a simple beach accessible only by boat, with some huts and a few cheap restaurants. The area is popular with rock climbers, and I spent half my time climbing and half my time developing my thesis. It was a very fun and productive time. By hiking over a headland, or wading around at low tide, I could get to another beach, West Railay, which had restaurants with wifi. From here I could video-conference with my PhD advisor occasionally. I had brought a small netbook, which allowed me to read PDFs and access the internet. It was in this isolated, distraction-free environment that I fleshed out most of the ideas in Section 4 and Section 5.

Or at least the seeds were planted. Upon returning to Seattle, I spent almost a year cleaning up, improving, and writing up these results (among others), which came to constitute parts of Chapters 2 and 4 of my thesis [**Wol12**]. This was less fun and more stressful – finding subtleties and fixing mistakes in a race to the finish.

The more general results of Section 3 didn't arise until the fall of 2012. I was hired by the University of Western Ontario, on a postdoc with Dan Christensen. But we spent the year as visitors at the Instituto Superior Técnico in Lisbon, Portugal. As I worked to generalize and clean up the results for publication, I was able to relax again and gain some perspective. The idea emerged to examine the relationship between the Bousfield lattice of a quotient and the quotient of a Bousfield lattice. The results concerning square-zero objects (Prop. 2.7 to Cor. 2.9) arose during this time, as well. Finally, the paper was submitted in January 2013.

3.2. Process of development. This section tells the story of the process of development of the ideas and results. The approach is chronological and more conceptual.

I believe the first step was, in spring 2011, rereading Weibel's book on homological algebra [**Wei94**]. The projection formula appears there, in Section 10.8.1, for derived categories of bounded complexes. I wondered if this would extend to unbounded complexes. When I convinced myself it did, I had Key Idea 2.4: each ring map $f : R \to S$ induces adjoint functors on $D(R)$ and $D(S)$, and these might extend to maps between Bousfield lattices. I pursued this direction, and it yielded. The adjoint functors $f_\bullet$ and $f^\bullet$ are so nice, it seemed like a good direction to pursue further. There was enough detail to get my hands on explicit computations, but also enough formal results to give the feeling that this would lead somewhere.

On the one hand, I collected results on $f_\bullet$ and $f^\bullet$, for example that $f_\bullet$ preserves compact objects, and the projection formula holds. On the other hand, I established the maps on the level of Bousfield lattices, and investigated how they treated the substructure BA and DL. The results of Section 4.1 and 4.2 emerged thus.

It became clear, through lots of fiddling, that a nice setting to work in would be one where $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X. With this assumption, it was straightforward to prove most of the results in Section 4.3. (I only figured out Theorem 4.18 in the fall of 2012, as I was refining the results for publication.)

(Much of my attention at this point was focused on using these results to connect what is known about Noetherian rings to the non-Noetherian case. For example, if $f : R \to S$ is a surjective ring map and S is Noetherian, the condition in Section 4.3 holds and those results apply. This material appeared in my thesis, but not the paper [**Wol14**].)

During this same time, I returned to [**DP08**] and the question of what results therein could be extended to $D(\Lambda_{\mathbb{Z}_{(p)}})$. In a very careful line-by-line rereading of [**DP08**], I reworked each proof to show what extended, didn't extend, or extended in some altered form. Having these two directions of research simultaneously in my mind, creating a synergy, it was only a matter of time before I had Key Idea 2.5: to use the maps $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ and $h : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$ to connect $D(\Lambda_{\mathbb{Z}_{(p)}})$ to $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Q}})$, which are covered by [**DP08**].

What followed was a period of significantly ugly computations, as I tried to figure out precisely what these functors were doing. Once I figured out that $g_\bullet$ satisfied the condition $\langle g_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X, I felt like I was getting somewhere. I noticed that something was going on, but it wasn't until the fall of 2011, back in Seattle, that I came to suspect there was a smashing localization functor involved. I was computing, and re-computing, facts such as those contained in Proposition 5.8. Finally I noticed these Bousfield classes were behaving like those that come from smashing localizations.

Once I had that idea, Key Idea 2.6, everything clarified. I knew the Iyengar-Krause result [**IK13**, Prop. 6.12] implied Key Idea 2.7: every smashing localization causes a splitting of the Bousfield lattice. (At the time, I found this result remarkable, surprising, and bizarrely under-expressed in their then-preprint.) It seemed that I was going to be able to prove a splitting of $\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})$.

At first it seemed that all my partial results, about maps from quotients of lattices, in Section 4, were going to be subsumed by this stronger result. This turned out not to be the case. The splitting only gives a result like Theorem 5.14, which splits $\mathsf{BL}(\mathsf{T})$ in terms of Bousfield lattices of subcategories. One must then use the work of Sections 2, 3, and 4 to tighten this result and in fact get a copy of $\mathsf{BL}(\Lambda_k)$ inside of $\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})$, as given in Corollary 5.17.

It was a natural next step to try to show that the splitting of Corollary 5.17 descended to a splitting of the distributive lattices DL and the Boolean algebras BA. But this revealed some unforeseen subtleties: what does it mean to be complemented in $\mathsf{BL}(\mathsf{loc}(h^\bullet\Lambda_{\mathbb{Q}}))$, where the maximum class is $\langle h^\bullet\Lambda_{\mathbb{Q}}\rangle$ and not $\langle \mathbb{1}\rangle$?

My attention then focused on trying to make sense of something like $\mathsf{BL}(\mathsf{loc}(h^\bullet\Lambda_{\mathbb{Q}}))$. This was during the fall of 2012. From Key Ideas 2.1 and 2.2, I knew that this Bousfield lattice needed to be reckoned with, but to date no one had considered $\mathsf{BL}(\mathsf{T})$ when $\mathbb{1} \notin \mathsf{T}$. It was necessary to revisit many of the well-known facts about Bousfield lattices, and carefully reassess them in the case of a proper subcategory.

The product of this reassessment is the treatment of Bousfield lattices in Section 2. I had to decide what it should mean, for example, to be complemented, or to be in BA. This involved lots of back-and-forth: establishing a tentative definition, seeing if this behaved like I thought it ought to, and checking it reduced to the familiar notions when necessary. For example, Definition 2.4(3): BA is the collection of Bousfield classes in DL that are complemented and have a complement in DL. This wouldn't be the first guess for a definition of BA, but this is the right one, in the end.

During the fall of 2012, I had been thinking of new places to look for Bousfield lattices, thanks to Key Idea 2.1. One such place is Verdier quotients, i.e. categories of locals. This led to Key Idea 2.3. It was straightforward to develop the results of Section 3, following my nose. There is a subtlety about what it means to be a lattice isomorphism, but this turns out to be a good thing. Corollary 3.3 is a nice result relating the lattice of a quotient and the quotient of a lattice. It was fun and satisfying to work through examples, and show that for my favorite weird categories, $D(\Lambda_k)$ and $\mathcal{S}$, this is *not* a lattice isomorphism (Proposition 3.5).

(I remember a fierce days-long episode of clarity on these last results, beginning January 1, 2013 after returning from a night spent in isolation with the full moon, in a cave on a small hard-to-get-to Portuguese beach.)

The final stage involved zooming out results as much as possible, and putting them in as general a framework as I could. This meant careful attention to well-generatedness, in Section 3, and careful attention to grading issues, in Section 4. For this intricate work, and writing up the Introduction and background material, alas, I needed to be inside, away from trains and beaches.

4. Colloquial summary

A ring, call it R, is a basic type of mathematical object, one that is introduced to students in their very first undergraduate course of "higher" math (that is, math based on theorems and proofs). A ring is just a set with a sensible notion of "addition" and "multiplication" on its elements, like the set of integers or the set of fractions. The derived category $D(R)$ of a ring R is an elaborate construction, through several layers of abstraction and added structure. Students usually encounter this notion in the first or second year of math grad school. Sometimes I think of the metaphor: rings are like fruit, and derived categories are like pies made from those fruit. Apples yield apple pie; peaches yield peach pie. We can understand a fruit by studying the type of pie it yields.

To be slightly more honest, I go one step further and make a lattice out of each derived category. This is called the Bousfield lattice, and mathematical lattices actually bear some resemblance to the lattices we encounter in real life. It's hard to see how a fruit pie turns into a picket fence, but take my word for it.

The results in the paper [**Wol14**] start out very general, and become more specific. The goal, or motivation, is to understand the (Bousfield lattice of the) derived category of a certain strange ring $\Lambda_{\mathbb{Z}_{(p)}}$. A very similar ring $\Lambda_{\mathbb{F}_p}$ was studied in the paper [**DP08**]. These rings are so similar that you might think of them as two varieties of apple.

In order to connect what is known about $D(\Lambda_{\mathbb{F}_p})$ with $D(\Lambda_{\mathbb{Z}_{(p)}})$, we need to explicitly relate $\Lambda_{\mathbb{F}_p}$ and $\Lambda_{\mathbb{Z}_{(p)}}$. The way this is done is with a function g from $\Lambda_{\mathbb{Z}_{(p)}}$ to $\Lambda_{\mathbb{F}_p}$. In Section 4 of [**Wol14**], I show that such a function g will induce a function $g_\bullet$ between the Bousfield lattices of derived categories. This is how mathematicians make precise, and computational, the idea: relate $\Lambda_{\mathbb{F}_p}$ to $\Lambda_{\mathbb{Z}_{(p)}}$, and use this to relate $D(\Lambda_{\mathbb{F}_p})$ to $D(\Lambda_{\mathbb{Z}_{(p)}})$.

Just as in real life, a common way to understand something in math is to try to break it into smaller pieces that are better understood. For example, 20 is the product of 4 and 5. The main results of [**Wol14**] do exactly this: the lattice built from the $\Lambda_{\mathbb{Z}_{(p)}}$ "pie" is nothing more, and nothing less, than a "product" of the lattice built from the $\Lambda_{\mathbb{F}_p}$ pie, with the lattice built from a third variety of apple $\Lambda_{\mathbb{Q}}$. More honestly, it turns out we don't use the whole $\Lambda_{\mathbb{Q}}$ pie, but a subcategory of it, like a slice! Fortunately, the case of $\Lambda_{\mathbb{Q}}$ and $D(\Lambda_{\mathbb{Q}})$ is also understood through the work previously done in [**DP08**].

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