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A user's guide: Bousfield lattices of non-Noetherian rings

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1. Key insights and central organizing principles

Conceptually, the starting point for [**Wol14**] is the paper [**DP08**] by Dwyer and Palmieri, studying the Bousfield lattice of the derived category of a certain non-Noetherian ring Λ_k, where k is a countable field.

As explained in the Introduction in [**Wol14**], and in the Philosophy section in [**DP08**], there are formal parallels between this lattice, and the Bousfield lattice of the p-local stable homotopy category $\mathcal{S}$, which arises in topology. The latter is very bizarre and hard to grasp, and derived categories of non-Noetherian rings offer a somewhat simpler setting. In the analogy between topology and algebra, it becomes natural to consider Dwyer and Palmieri's ring Λ_k, but with the p-local integers $\mathbb{Z}_{(p)}$ replacing k. This is $\Lambda_{\mathbb{Z}_{(p)}}$.

Then the obvious, and somewhat uninspired, research question is: What results of [**DP08**] carry over to $D(\Lambda_{\mathbb{Z}_{(p)}})$?

The results in the paper [**Wol14**] build towards a fairly decent answer to this question. Various tools and machinery are developed along the way. For each such tool, it is natural to try to state and prove results in as much generality as possible. Thus the paper progresses from general to specific, in three distinct phases, Sections 3 – 5. This is the literal order of exposition, but also a helpful conceptual roadmap. Each phase has a very different conceptual flavor.

1.1. Intro and Section 2. Before discussing Sections 3 – 5, it's worth saying some things about the first two sections. The Introduction and Section 2 were intended to be a careful, somewhat thorough introduction to Bousfield lattices: what they are, why they are interesting, and examples of what we know about them. This stemmed from two insights.

Key Idea 1.1. [**IK13**, Thm. 3.1] *In any well generated tensor triangulated category, there is a set of Bousfield classes (rather than a proper class), and thus a Bousfield lattice.*

Key Idea 1.2. *The proof of* [**IK13**, Thm. 3.1] *also applies to well generated localizing tensor ideals of a tensor triangulated category.*

The first idea suggests that Bousfield lattices show up in more places than previously realized, and so deserve a unified treatment. But the second idea is more exciting: if $\mathsf{S} \subseteq \mathsf{T}$ is such a localizing tensor ideal, then we can talk about its Bousfield lattice $\mathsf{BL}(\mathsf{S})$. But now the tensor unit may not be in S! This causes many previous known facts about Bousfield lattices to break. For example, it is no longer the case that $\langle X \rangle = \langle 0 \rangle$ implies $X = 0$, the maximum Bousfield class is no longer $\langle \mathbb{1} \rangle$, and complements are no longer unique.

It becomes necessary to carefully revisit what we know about Bousfield classes, and treat them in this more general light. This is what Section 2 is about. I also included a long subsection on examples, to gather facts that I thought were illuminating in getting a sense of what Bousfield lattices can be like.

1.2. Section 3.

The key idea in Section 3 is the following.

Key Idea 1.3. *The Verdier quotient T/S of a well generated tensor triangulated category T is also well generated. So it has a Bousfield lattice. What is it? There is a quotient functor $\pi : \mathsf{T} \to \mathsf{T}/\mathsf{S}$; how does this connect the two Bousfield lattices?*

Sure enough, the quotient functor induces a map between the Bousfield lattices. You might expect this is trivial, but in fact I think it's only true if $\mathsf{S} = \langle Z \rangle$ is a Bousfield class! In other words, you can't just mod out by anything, it has to be a Bousfield class. I took that to mean that at least something non-trivial is going on.

Section 3 runs with this idea. Lemma 3.1 gives an onto join-morphism. When you have an onto map, you try to take the kernel and sharpen it. Thus it's natural to consider the quotients of $\mathsf{BL}(\mathsf{T})$. And then you have the following clever question.

Question 1.4. *What is the relationship between the Bousfield lattice of a quotient, and the quotient of a Bousfield lattice?*

Most of the results in Section 3 center on this question. Proposition 3.2 seems to give a good answer, but really points out a key weirdness about lattices. The map in Proposition 3.2 is onto, has trivial kernel, and preserves arbitrary joins. But this isn't enough for a lattice isomorphism! Proposition 3.5 gives two examples where this is *not* a lattice isomorphism. On the other hand, in Corollaries 3.3 and 3.4, we have two cases where there *is* an isomorphism. The more important one is Corollary 3.3. This is a more-or-less satisfying answer to Question 1.4.

1.3. Section 4. Section 4 is centered on trying to understand how a map of rings $f : R \to S$ can relate the Bousfield lattices $\mathsf{BL}(D(R))$ and $\mathsf{BL}(D(S))$. Some results are in complete generality, true for any ring map f. But the more interesting ones are in Section 4.3, when we add a hypothesis.

The reader is quickly confronted with a severe warning, and it's worth pointing out that this is *not* a key insight or central idea. The issue is simply this: the rings Λ_k and $\Lambda_{\mathbb{Z}_{(p)}}$ are graded, and then their derived categories are bigraded, and this is a little unusual. I wanted my statements and results in this section to hold for them. But I also wanted to talk about the rings that more people cared about, i.e. ungraded rings. Since all the proofs applied in both cases, the most streamlined way I could think to explain things was by laying out the warnings.

KEY IDEA 1.5. *A ring map $R \xrightarrow{f} S$ induces maps $f_* : \mathsf{Mod}\text{-}R \rightleftarrows \mathsf{Mod}\text{-}S : f^*$, which descend to derived categories. These are very, very nice functors. For example, the projection formula holds.*

Section 4.1 is full of helpful properties of these functors. The projection formula is probably the single most useful result in the whole paper, and gets used over and over in proofs. Projection formulas like this appear in many other contexts; this setup is sometimes called the "Grothendieck context".

The results in Section 4.2 are mostly partial and unsatisfying, only precursors to their improvements in Section 4.3. The condition there – $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X – is a strong and useful one. It's essentially a surjectivity condition, and so I expect it to hold in situations where R is "big" and S is "small". (For example, in my thesis I show that this condition is satisfied when $f : R \to S$ is a surjection onto a Noetherian ring.)

It is also a somewhat arbitrary condition. A more complete treatment would have considered other types of ring maps (injective?), or other conditions. But in the paper there's only this one condition considered; it is the one that yields nice results (of these I think Theorem 4.18 is the nicest), and applies in Section 5.

1.4. Section 5. Finally in Section 5 we have specialized to the original question: what can we say about the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$?

KEY IDEA 1.6. *The ring $\Lambda_{\mathbb{Z}_{(p)}}$ is closely related to the two rings $\Lambda_{\mathbb{F}_p}$ and $\Lambda_{\mathbb{Q}}$, which are covered in the Dwyer-Palmieri paper* [**DP08**] *because $\mathbb{F}_p$ and $\mathbb{Q}$ are countable fields.*

For example, we have the projection $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$. Since it's surjective, we can ask if the condition of Section 4.3 holds. The first order of business is Proposition 5.3, showing that it does. For $g_\bullet$ at least; my guess is that $h_\bullet$ does not satisfy the condition of Section 4.3, but I was unable to prove or disprove it. Thus we can apply all the results from Section 4, using $g_\bullet$. This gives various lattice maps, and lattice isomorphisms with quotients. But the strongest results come from two realizations.

Key Idea 1.7. *The three derived categories* $D(\Lambda_{\mathbb{Z}_{(p)}})$*,* $D(\Lambda_{\mathbb{F}_p})$*, and* $D(\Lambda_{\mathbb{Q}})$ *are related by a smashing localization functor* $L : D(\Lambda_{\mathbb{Z}_{(p)}}) \to D(\Lambda_{\mathbb{Z}_{(p)}})$*, finite localization away from* $\mathsf{th}(g^{\bullet}\Lambda_{\mathbb{F}_p})$*.*

Key Idea 1.8. *Any smashing localization causes a splitting of the Bousfield lattice into a product of lattices.*

From the first realization we can perform many computations, and Proposition 5.8 gives a clear picture of what's equal to what. But the second realization, laid out in Lemmas 5.11 and 5.12, Proposition 5.13, and Theorem 5.14, is more structurally powerful. Combining the two, we get a very explicit and satisfying splitting of the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$.

At the end, Corollary 5.18 is a nice non-trivial application of the theory developed in Section 2, on Bousfield lattices of localizing tensor ideals that don't contain the tensor unit. Corollary 5.19 is only included for the record; cardinality calculations are useful qualitative facts to have available.

References

[DP08] W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology Homotopy Appl. **10** (2008), no. 1, 413–436.

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