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A user's guide: Bousfield lattices of non-Noetherian rings

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2. Metaphors and imagery

The central objects of the paper are Bousfield lattices of derived categories of non-Noetherian rings, and this triple-layer idea has three levels of imagery.

2.1. Bousfield lattice. Bousfield lattices are quite visual and explicit in my mind. They feel simpler – simpler than the categories they're built out of, at least. Considering "the Bousfield lattice of..." involves a mental stepping-above, moving up a level. This is usually experienced as moving up in my visual field, or less commonly lifting up off the page. Each symbol $\mathsf{BL}(-)$ evokes this cleaner, more transparent feeling. Bousfield lattices are posets, with a bottom and a top. I see vertices and lines connecting them, indicating the partial order, moving from bottom to top. The join of a set of elements is obviously positioned above them somewhere, the meet below. These images are obvious and were probably learned in a classroom or from a textbook. With Bousfield lattices, the joins – even infinite joins – are easy to compute, and show up frequently.

The smash operation also appears below its arguments, further down than the meet. It feels stable and comforting. For the smash commutes with coproducts, hence arbitrary joins. It's not too hard to compute this operation, and manipulate it efficiently. The meet, on the other hand, is hard to pin down, is ideally avoided in computations, and exists in a visual position but a vague one.

While computing with Bousfield classes, as in Section 2 of the paper, there's a fair amount of symbol manipulation. This is very formal; I know various manipulations and formulas, and can apply them when appropriate. They wait behind me, out of my vision, but come forward when something on the paper starts to ring a bell. It's almost a form of pattern recognition; at each step the calculations displayed before me are suggesting operations to perform. I can only go forward about two steps in my head, without writing things down.

If prompted to contemplate the structure of the Bousfield lattice, I immediately see the statement $\mathsf{BA} \subseteq \mathsf{DL} \subseteq \mathsf{BL}$. I know that in many cases they are all equal, but in the cases I'm mostly fascinated by they aren't. There's a big mysterious gap between BL and DL, and floating there are the square-zero objects, represented by the notorious $I \wedge I = 0$. Both DL and BA have the feel of nicer, more structured lattices, albeit sitting inside BL awkwardly. The objects in BA appear to me in pairs, since these are the complemented classes. There is a special sub-collection among them, the pairs $\{\langle C\mathbb{1}\rangle, \langle L\mathbb{1}\rangle\}$ arising from smashing localization functors.

Once on a hike in the rain, I looked at the drops falling in a puddle on the trail, and I thought I saw something complex enough, rich enough, to be "what a Bousfield lattice looks like". Not in a way that "I understand BL now", but rather I understood it could be understood, even if it were so complex that that understanding wouldn't take a form we had yet conceived of. Although sadly I won't be able to describe the image, I can sort of see it in my mind's eye, and I evoke this puddle experience (because it was as much about the image as the rain, the scenery, and the hotspring I was leaving) when I need to remember what I'm dealing with and why.

2.2. Derived categories. The key to understanding and being able to work with derived categories is establishing a strong hierarchy of conceptual levels. Here I'll just talk about derived categories of rings. A ring R gives a module category $\mathsf{Mod}\text{-}R$. This needs to be treated as a black box, a nice abelian category where almost everything works as we would want it to. The category of chain complexes $Ch(R)$ has a flavor completely of its own – its own obvious visual images of complexes of modules, its own body of results and computations. Every time I inhabit this category, I see commutative diagrams between chain complexes, and computations extracting information from these.

But to pass to $D(R)$, I strongly suggest reorienting. Forget chain complexes as much as possible. This category $D(R)$ has a host of formal properties, and it is most helpful to clean the plate and start from scratch with these. The approach of axiomatic stable homotopy theory obviously encourages this. Over time, a map of objects $X \to Y$ becomes exactly that – just two symbols connected by an arrow. No chain complex representatives, no modules, no rings acting. I know there's a tensor operation – heuristically a multiplication – with a unit. There are coproducts, which I think of as an addition operation. The triangles are fun. They take some getting used to, but after enough practice seeing how one uses them in computations, they become powerful and enjoyable tools. Every morphism fits into such a triangle. Morphisms between triangles behave nicely. The ability to turn a triangle into a long exact sequence of homology groups imbues the triangles with a strength and sharpness.

The way to think about derived categories is to work with these formal properties as much as possible. An object is just an object. But it inevitably becomes necessary to dig deeper, into the structure of the particular axiomatic stable homotopy category you are working with. And so you must choose a representative, write down a chain complex, and stare at it. Knowing when to pass from the $D(R)$ level to the $Ch(R)$ level, and back up, is an art.

I've found it helpful to nurture the distinct conceptual flavors, and atmospheres, of the two levels $D(R)$ and $Ch(R)$. Each has developed its own ecosystem in my mind. This helps me to immerse myself in the subtleties and functionalities of one level, with less distraction and confusion. Moving between levels is a discrete cognitive act, and then I'm in a new environment, with its own flavor. The ability to nurture this separation of ideas, and navigate between, seems crucial to understanding abstract mathematics and harnessing its power.

To give an example, consider Definition 5.6. The inclusion $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$ of the p-local integers into the rationals evokes basic abstract algebra imagery: these rings are sets of elements with structure; ring maps are somewhat rigid; we can imagine writing out fractions a/b. Then I contemplate that this induces a morphism $\Lambda_{\mathbb{Z}_{(p)}} \to h^\bullet\Lambda_{\mathbb{Q}}$ in $D(\Lambda_{\mathbb{Z}_{(p)}})$. First I see the inclusion passing to an inclusion $\Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$ (see comments below). Then I imagine these as chain complexes, concentrated in degree zero, in $Ch(\Lambda_{\mathbb{Z}_{(p)}})$. You can see this clearly: two infinite complexes, mostly zeros, with an arrow mapping from $\Lambda_{\mathbb{Z}_{(p)}}$ to $\Lambda_{\mathbb{Q}}$. This arrow, a $\mathbb{Z}_{(p)}$-module map, still prompts me to think about the elements inside these modules. But then we switch to the derived category level. I see the target complex as a representative of $h^\bullet\Lambda_{\mathbb{Q}}$, and suddenly there is only two symbols connected by an arrow: $\Lambda_{\mathbb{Z}_{(p)}} \to h^\bullet\Lambda_{\mathbb{Q}}$. The symbol $h^\bullet$ reminds me this is a high-level functor, between derived categories that are themselves now zoomed-out black boxes. Now, in this triangulated world, it is a simple step to extend this map to a triangle, and define F as the fiber. At this point in my mind, F really exists just as a symbol completing a triangle.

2.3. Non-Noetherian rings. The paper is very notation-heavy, and I think there's a real danger of getting lost by the sub-subscripts. At no point do I use the substructure of, say $\Lambda_{\mathbb{Z}_{(p)}}$, to prove anything; there are no proofs manipulating the x_i's or n_i's. For this reason, I think it's helpful to mostly forget about what $\Lambda_{\mathbb{Z}_{(p)}}$ is, except for being a non-Noetherian polynomial $\mathbb{Z}_{(p)}$-algebra. Then $\Lambda_{\mathbb{Q}}$ is the same, but with a $\mathbb{Q}$ everywhere that $\Lambda_{\mathbb{Z}_{(p)}}$ has a $\mathbb{Z}_{(p)}$, and likewise $\Lambda_{\mathbb{F}_p}$. Most of my intuition about how $\mathbb{Z}_{(p)}$, $\mathbb{F}_p$, and $\mathbb{Q}$ relate gets carried over to $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$. This certainly helps in suggesting to me what might be true about the latter; proving these statements requires care, of course.

I never really use the non-Noetherian nature of these rings, except insofar as I build on results in [**DP08**]. (In that paper, they certainly compute with x_i's.) The key point is that Noetherian rings are too nice. The Bousfield lattice of the derived category of a Noetherian ring is completely understood, and is too nice (in my opinion). These rings $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$ are some of the simplest non-Noetherian rings to consider.

2.4. Other concepts. Next I'll describe how I think about a few of the other ideas that show up in the paper.

2.4.1. *Tensor triangulated categories.* I discussed this above, when talking about the formal properties of $D(R)$ and working at that level. Since this formal structure is all that tensor triangulated categories have, by definition, one has no choice but to use that and only that. But even when proving things about tensor triangulated categories, I find it useful to occasionally "project down" to one of the colorful examples, like spectra or $D(R)$. This might suggest intuition about what to expect to be true. But it also nicely enriches the dry category-theoretical reasoning, with extra flavor and (unnecessary but still attractive) details.

Well generated categories are hard to grasp. Their definition is complicated, and it takes a long time to build intuition. I find it best to treat them as "the right class of triangulated categories". That is, they are closed under taking subcategories and quotients, more or less. The more specialized class of compactly generated triangulated categories is not. Most results about well generated categories that I feel I understand, are natural generalizations of results on compactly generated categories. So the essence is the same: with arbitrary coproducts, the full category is "large"; we ask for a smaller generating set, which helps us do "small" things; in good situations results about small things can be generalized to the whole large category. For compactly generated categories, the small things are quite small; in well generated categories we allow for less-small things. Thus a compactly generated category is the same as an $\aleph_0$-well generated category, but going less-small we consider regular cardinals larger than $\aleph_0$.

2.4.2. *Lattice quotients and products.* I don't feel that I have good metaphors and imagery for these concepts, and this prevents me from feeling like I understand what's going on in some of my results about them.

I certainly don't have much intuition about a lattice quotient, although it is reassuring to remember that $\mathsf{L}/a\!\downarrow \,\cong a\!\uparrow$ (see Section 2.3 of [**Wol14**] for definitions). The sublattice $a\!\uparrow$ invokes a strong, and obvious, visual image. Otherwise, I treat any object $\mathsf{BL}(\mathsf{T})/J$ quite formally, going back to its definition. Similarly, I am befuddled about the Bousfield lattice splitting induced by smashing localizations, described in Theorem 5.14. I feel like I should be able to "see" what such a product looks like, but haven't quite gotten there. Results like Proposition 5.13 certainly help.

2.4.3. *Bousfield lattices of subcategories.* Lattices of the form $\mathsf{BL}(\mathsf{loc}(X))$, where $\mathsf{loc}(X) \subseteq \mathsf{T}$, seem mysterious, in a good way. Section 2 facts, like that complements aren't unique, and the maximum is $\langle X \rangle$, are intriguing. I'm comfortable working in this Bousfield lattice, slowly, but find it confusing to tease out the relationship between this lattice and the full Bousfield lattice $\mathsf{BL}(\mathsf{T})$. Definition 5.10 and Lemma 5.12 were inspired by my attempt to clarify this. Fortunately, in the case of a smashing localization, as in Section 5, these are well-behaved.

2.4.4. *Localization.* Localization functors don't feature too prominently in this paper, but when they show up, in Section 5, they are very powerful tools. It took me a long time, and lots of practice, to get comfortable with localizations. Maybe all that I will say here is, I've found it useful to alternate between the localization perspective and the Verdier quotient perspective.

Every localization on T is determined by its acyclics, a nice reliable localizing subcategory S. The existence of such a localization is equivalent to the Verdier quotient T/S having Hom sets, i.e. being a triangulated category on its own. This category T/S is easy to think about, formally at least: it is a "projection" of T where everything in S (and maps that factor through an object in S) are killed. (For years I fought the murderous imagery, but reluctantly am adopting it; there's no concise alternative.) The quotient map $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ has a right adjoint, and the localization on T is just the composition of these.

One key idea is this: that right adjoint induces an equivalence between T/S and the image of the localization functor, i.e. the category of locals. This equivalence gives the category of locals its own tensor triangulated structure. So when thinking about local objects, I alternate between considering quotient objects in T/S, and considering honest objects of T but with their own tensor product and coproducts. In the case of a smashing localization – a very, very nice type of localization – these are the same as in T.

2.4.5. *The functors $f_\bullet$ and $f^\bullet$.* The adjoint pair $f_\bullet : D(R) \rightleftarrows D(S) : f^\bullet$ is the central idea of Section 4, and has a lot of associated imagery and metaphors. Both $f_\bullet$ and $f^\bullet$ feel like good, well-behaved, nice functors, that have good properties that one can get a handle on. Computing with them is straightforward. There is a concrete image of $D(R)$ on the left, $D(S)$ on the right, and the two functors moving objects and morphisms back and forth.

The left adjoint $f_\bullet$, which sends R to S and preserves compact objects, seems to be the nicer of the two, a more handy machine. It's more structural, like an architect. If it were colored, it would be green or blue. The right adjoint $f^\bullet$, which is just a forgetful functor, acts by taking an object in $D(S)$ and simply plopping it down into $D(R)$. The consequence is more delicate, and has more personality. I would say $f^\bullet$ is peaceful and quiet, maybe off-white, a light yellow, or gentle gray.

The fact that both $f_\bullet$ and $f^\bullet$ give order-preserving maps on Bousfield lattices has a definite visual image. I see $\langle X\rangle \leq \langle Y\rangle$, and then I see $f_\bullet\langle X\rangle \leq f_\bullet\langle Y\rangle$ transported to the right, in the same spatial orientation but maybe stretched or translated.

The projection formula was mostly absorbed, over time, as one of the various options for manipulation of formulas, as discussed above when discussing Bousfield lattices. It is so important and useful, though, that it's one of the first patterns I try to apply and take advantage of. During the most focused time of working on this paper, I think the projection formula, and the Corollary 4.5 version of it, became so saturated in my thoughts that I often performed Bousfield class computations in parallel in my mind – one version in $\mathsf{BL}(R)$ and one version in $\mathsf{BL}(S)$. The projection formula is a way of moving back and forth, and I reached the point where I maintained both lines of reasoning simultaneously.

As mentioned previously, the condition in Section 4.3 – that $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X – strikes me as a surjectivity condition, heuristically. In this context, I visualize $\mathsf{BL}(R)$ larger than $\mathsf{BL}(S)$, and mapping onto it, with $\mathsf{DL}(R)$ and $\mathsf{BA}(R)$ mapping onto their counterparts. The functor $f^\bullet$ embeds $\mathsf{BL}(S)$ (and $\mathsf{DL}(S)$ and $\mathsf{BA}(S)$) into $\mathsf{BL}(R)$ in an intriguing way. For example, the statement $\langle f^\bullet S\rangle \vee \langle M_f\rangle = \langle R\rangle$ is just stating that $\langle S\rangle$ in $\mathsf{BL}(S)$ (which is complemented there by $\langle 0\rangle$) gets sent to a complemented class in $\mathsf{BL}(R)$ (in which case its complement is now $\langle M_f\rangle$).

References

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