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A user's guide: Bousfield lattices of non-Noetherian rings

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3. Story of the development

I have broken the story of development into two parts: logistical and conceptual.

3.1. Where and when the ideas arose. This section sets the stage for the next, by describing the context of where and when the ideas arose. The details are logistical and psychological.

The idea to investigate $D(\Lambda_{\mathbb{Z}_{(p)}})$ came during the summer of 2009, while I was riding the Trans-Siberian railway between the east coast of Russia and Irkutsk. It was the summer after my third year of math graduate school; my newly established PhD advisor suggested I read [**Mar83**] and [**HPS97**] to become familiar with this axiomatic approach to tensor triangulated categories and stable homotopy categories. I had recently read [**HS98**], [**HP99**] and [**DP08**], so my mind was primed for drawing a stronger connection between the p-local stable homotopy category $\mathcal{S}$ and the derived category $D(\Lambda_k)$. But it wasn't that exciting of a research question, so I didn't pursue it very far.

In the spring of 2011 I took a leave of absence from grad school, and went to Thailand. My base was Tonsai – a simple beach accessible only by boat, with some huts and a few cheap restaurants. The area is popular with rock climbers, and I spent half my time climbing and half my time developing my thesis. It was a very fun and productive time. By hiking over a headland, or wading around at low tide, I could get to another beach, West Railay, which had restaurants with wifi. From here I could video-conference with my PhD advisor occasionally. I had brought a small netbook, which allowed me to read PDFs and access the internet. It was in this isolated, distraction-free environment that I fleshed out most of the ideas in Section 4 and Section 5.

Or at least the seeds were planted. Upon returning to Seattle, I spent almost a year cleaning up, improving, and writing up these results (among others), which came to constitute parts of Chapters 2 and 4 of my thesis [**Wol12**]. This was less fun and more stressful – finding subtleties and fixing mistakes in a race to the finish.

The more general results of Section 3 didn't arise until the fall of 2012. I was hired by the University of Western Ontario, on a postdoc with Dan Christensen. But we spent the year as visitors at the Instituto Superior Técnico in Lisbon, Portugal. As I worked to generalize and clean up the results for publication, I was able to relax again and gain some perspective. The idea emerged to examine the relationship between the Bousfield lattice of a quotient and the quotient of a Bousfield lattice. The results concerning square-zero objects (Prop. 2.7 to Cor. 2.9) arose during this time, as well. Finally, the paper was submitted in January 2013.

3.2. Process of development. This section tells the story of the process of development of the ideas and results. The approach is chronological and more conceptual.

I believe the first step was, in spring 2011, rereading Weibel's book on homological algebra [**Wei94**]. The projection formula appears there, in Section 10.8.1, for derived categories of bounded complexes. I wondered if this would extend to unbounded complexes. When I convinced myself it did, I had Key Idea 2.4: each ring map $f : R \to S$ induces adjoint functors on $D(R)$ and $D(S)$, and these might extend to maps between Bousfield lattices. I pursued this direction, and it yielded. The adjoint functors $f_\bullet$ and $f^\bullet$ are so nice, it seemed like a good direction to pursue further. There was enough detail to get my hands on explicit computations, but also enough formal results to give the feeling that this would lead somewhere.

On the one hand, I collected results on $f_\bullet$ and $f^\bullet$, for example that $f_\bullet$ preserves compact objects, and the projection formula holds. On the other hand, I established the maps on the level of Bousfield lattices, and investigated how they treated the substructure BA and DL. The results of Section 4.1 and 4.2 emerged thus.

It became clear, through lots of fiddling, that a nice setting to work in would be one where $\langle f_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X. With this assumption, it was straightforward to prove most of the results in Section 4.3. (I only figured out Theorem 4.18 in the fall of 2012, as I was refining the results for publication.)

(Much of my attention at this point was focused on using these results to connect what is known about Noetherian rings to the non-Noetherian case. For example, if $f : R \to S$ is a surjective ring map and S is Noetherian, the condition in Section 4.3 holds and those results apply. This material appeared in my thesis, but not the paper [**Wol14**].)

During this same time, I returned to [**DP08**] and the question of what results therein could be extended to $D(\Lambda_{\mathbb{Z}_{(p)}})$. In a very careful line-by-line rereading of [**DP08**], I reworked each proof to show what extended, didn't extend, or extended in some altered form. Having these two directions of research simultaneously in my mind, creating a synergy, it was only a matter of time before I had Key Idea 2.5: to use the maps $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ and $h : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$ to connect $D(\Lambda_{\mathbb{Z}_{(p)}})$ to $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Q}})$, which are covered by [**DP08**].

What followed was a period of significantly ugly computations, as I tried to figure out precisely what these functors were doing. Once I figured out that $g_\bullet$ satisfied the condition $\langle g_\bullet f^\bullet X\rangle = \langle X\rangle$ for all X, I felt like I was getting somewhere. I noticed that something was going on, but it wasn't until the fall of 2011, back in Seattle, that I came to suspect there was a smashing localization functor involved. I was computing, and re-computing, facts such as those contained in Proposition 5.8. Finally I noticed these Bousfield classes were behaving like those that come from smashing localizations.

Once I had that idea, Key Idea 2.6, everything clarified. I knew the Iyengar-Krause result [**IK13**, Prop. 6.12] implied Key Idea 2.7: every smashing localization causes a splitting of the Bousfield lattice. (At the time, I found this result remarkable, surprising, and bizarrely under-expressed in their then-preprint.) It seemed that I was going to be able to prove a splitting of $\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})$.

At first it seemed that all my partial results, about maps from quotients of lattices, in Section 4, were going to be subsumed by this stronger result. This turned out not to be the case. The splitting only gives a result like Theorem 5.14, which splits $\mathsf{BL}(\mathsf{T})$ in terms of Bousfield lattices of subcategories. One must then use the work of Sections 2, 3, and 4 to tighten this result and in fact get a copy of $\mathsf{BL}(\Lambda_k)$ inside of $\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}})$, as given in Corollary 5.17.

It was a natural next step to try to show that the splitting of Corollary 5.17 descended to a splitting of the distributive lattices DL and the Boolean algebras BA. But this revealed some unforeseen subtleties: what does it mean to be complemented in $\mathsf{BL}(\mathsf{loc}(h^\bullet\Lambda_{\mathbb{Q}}))$, where the maximum class is $\langle h^\bullet\Lambda_{\mathbb{Q}}\rangle$ and not $\langle \mathbb{1}\rangle$?

My attention then focused on trying to make sense of something like $\mathsf{BL}(\mathsf{loc}(h^\bullet\Lambda_{\mathbb{Q}}))$. This was during the fall of 2012. From Key Ideas 2.1 and 2.2, I knew that this Bousfield lattice needed to be reckoned with, but to date no one had considered $\mathsf{BL}(\mathsf{T})$ when $\mathbb{1} \notin \mathsf{T}$. It was necessary to revisit many of the well-known facts about Bousfield lattices, and carefully reassess them in the case of a proper subcategory.

The product of this reassessment is the treatment of Bousfield lattices in Section 2. I had to decide what it should mean, for example, to be complemented, or to be in BA. This involved lots of back-and-forth: establishing a tentative definition, seeing if this behaved like I thought it ought to, and checking it reduced to the familiar notions when necessary. For example, Definition 2.4(3):

BA is the collection of Bousfield classes in DL that are complemented and have a complement in DL. This wouldn't be the first guess for a definition of BA, but this is the right one, in the end.

During the fall of 2012, I had been thinking of new places to look for Bousfield lattices, thanks to Key Idea 2.1. One such place is Verdier quotients, i.e. categories of locals. This led to Key Idea 2.3. It was straightforward to develop the results of Section 3, following my nose. There is a subtlety about what it means to be a lattice isomorphism, but this turns out to be a good thing. Corollary 3.3 is a nice result relating the lattice of a quotient and the quotient of a lattice. It was fun and satisfying to work through examples, and show that for my favorite weird categories, $D(\Lambda_k)$ and $\mathcal{S}$, this is *not* a lattice isomorphism (Proposition 3.5).

(I remember a fierce days-long episode of clarity on these last results, beginning January 1, 2013 after returning from a night spent in isolation with the full moon, in a cave on a small hard-to-get-to Portuguese beach.)

The final stage involved zooming out results as much as possible, and putting them in as general a framework as I could. This meant careful attention to well-generatedness, in Section 3, and careful attention to grading issues, in Section 4. For this intricate work, and writing up the Introduction and background material, alas, I needed to be inside, away from trains and beaches.

References

[DP08] W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology Homotopy Appl. **10** (2008), no. 1, 413–436.

[HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610.

[HP99] Mark Hovey and John H. Palmieri, *The structure of the Bousfield lattice*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 175–196.

[IK13] Srikanth B. Iyengar and Henning Krause, *The Bousfield lattice of a triangulated category and stratification*, Math. Z. **273** (2013), no. 3-4, 1215–1241.

[Mar83] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29, North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wol12] F. Luke Wolcott, *A tensor-triangulated approach to derived categories of non-Noetherian rings*, Ph.D. thesis, University of Washington, 2012.

[Wol14] ———, *Bousfield lattices of non-Noetherian rings: some quotients and products*, Homology Homotopy Appl. **16** (2014), no. 2, 205–229.

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