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A user's guide: Bousfield lattices of non-Noetherian rings

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4. Colloquial summary

A ring, call it R, is a basic type of mathematical object, one that is introduced to students in their very first undergraduate course of "higher" math (that is, math based on theorems and proofs). A ring is just a set with a sensible notion of "addition" and "multiplication" on its elements, like the set of integers or the set of fractions. The derived category $D(R)$ of a ring R is an elaborate construction, through several layers of abstraction and added structure. Students usually encounter this notion in the first or second year of math grad school. Sometimes I think of the metaphor: rings are like fruit, and derived categories are like pies made from those fruit. Apples yield apple pie; peaches yield peach pie. We can understand a fruit by studying the type of pie it yields.

To be slightly more honest, I go one step further and make a lattice out of each derived category. This is called the Bousfield lattice, and mathematical lattices actually bear some resemblance to the lattices we encounter in real life. It's hard to see how a fruit pie turns into a picket fence, but take my word for it.

The results in the paper [**Wol14**] start out very general, and become more specific. The goal, or motivation, is to understand the (Bousfield lattice of the) derived category of a certain strange ring $\Lambda_{\mathbb{Z}_{(p)}}$. A very similar ring $\Lambda_{\mathbb{F}_p}$ was studied in the paper [**DP08**]. These rings are so similar that you might think of them as two varieties of apple.

In order to connect what is known about $D(\Lambda_{\mathbb{F}_p})$ with $D(\Lambda_{\mathbb{Z}_{(p)}})$, we need to explicitly relate $\Lambda_{\mathbb{F}_p}$ and $\Lambda_{\mathbb{Z}_{(p)}}$. The way this is done is with a function g from $\Lambda_{\mathbb{Z}_{(p)}}$ to $\Lambda_{\mathbb{F}_p}$. In Section 4 of [**Wol14**], I show that such a function g will induce a function $g_\bullet$ between the Bousfield lattices of derived categories. This is how mathematicians make precise, and computational, the idea: relate $\Lambda_{\mathbb{F}_p}$ to $\Lambda_{\mathbb{Z}_{(p)}}$, and use this to relate $D(\Lambda_{\mathbb{F}_p})$ to $D(\Lambda_{\mathbb{Z}_{(p)}})$.

Just as in real life, a common way to understand something in math is to try to break it into smaller pieces that are better understood. For example, 20 is the product of 4 and 5. The main results of [**Wol14**] do exactly this: the lattice built from the $\Lambda_{\mathbb{Z}_{(p)}}$ "pie" is nothing more, and nothing less, than a "product" of the lattice built from the $\Lambda_{\mathbb{F}_p}$ pie, with the lattice built from a third variety of apple $\Lambda_{\mathbb{Q}}$. More honestly, it turns out we don't use the whole $\Lambda_{\mathbb{Q}}$ pie, but a subcategory of it, like a slice! Fortunately, the case of $\Lambda_{\mathbb{Q}}$ and $D(\Lambda_{\mathbb{Q}})$ is also understood through the work previously done in [**DP08**].

References

[DP08] W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology Homotopy Appl. **10** (2008), no. 1, 413–436.

[Wol14] F. Luke Wolcott, *Bousfield lattices of non-Noetherian rings: some quotients and products*, Homology Homotopy Appl. **16** (2014), no. 2, 205–229.

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