Enchiridion: Mathematical User's Guides Volume 1 (October 2015)
mathusersguides.com

A user's guide: The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic p-groups

Carolyn Yarnall

1. Key ideas and central organizing principles

1.1. Background. In stable homotopy theory, we can regard Eilenberg-MacLane spectra as the fundamental "building blocks" of other spectra. This notion is embodied by the Postnikov tower which encapsulates how a spectrum is constructed from its individual homotopy groups. The key characteristic then of this particular filtration is that the fibers are indeed Eilenberg-MacLane spectra. However, in equivariant stable homotopy theory, that is, in the context of spectra with a group action, the role of Eilenberg-MacLane spectra is not so simple.

To begin looking into the equivariant setting, we must note that the homotopy "groups" of genuine equivariant spectra are not merely groups. These homotopy groups for a G-spectrum X come from the fixed points of equivariant maps from S^n to X. However, to get the full picture, instead of merely considering the G-fixed points, we must consider the H-fixed points for all subgroups H of G. The homotopy groups of X^H for all H also have maps between them and all this data fits together to form a *Mackey functor*.

One might expect then that the proper way to extend the Postinikov tower to the equivariant setting would be to construct a tower whose fibers were all Eilenberg-MacLane spectra associated to Mackey functors. However, this type of filtration does not end up being the most natural choice for classic equivariant spectra. What we really want is a filtration that uses representation suspensions of Eilenberg-MacLane spectra; this is the *slice filtration*. This filtration for a G-spectrum X is formed in a similar fashion to the Postnikov tower but instead of killing maps (i.e. formally inverting them up to homotopy) from ordinary spheres to X, we kill maps from so-called *slice cells*. The result is that while we still get a tower whose limit is equivalent to X and whose colimit is contractible, the fibers are often more complicated G-spectra that we refer to as *slices*.

In [**HHR09**] the slice filtration was shown to filter the complex cobordism spectrum in a nice way. However, when applied to other spectra, we can get some more complicated results. What becomes mysterious in considering a slice tower rather than a Postnikov tower is that often the homotopy of each layer of the slice filtration differs from the next in more than one dimension. Additionally, if we have an object that is n-slice, or capable of being the fiber in the nth layer of a tower, and we suspend it, it may not be $(n+1)$-slice. Thus, the slice tower does not commute with taking integer suspensions. To get a better idea of how suspension and the slice filtration interact, it would be nice to know the slice towers for all suspensions of simple objects. As a start, in [**Yar15**] we wanted the following:

Goal: Determine the slice tower for all positive integer suspensions of $H\underline{\mathbb{Z}}$, the Eilenberg-MacLane spectrum associated to the constant G-Mackey functor where G is a cyclic p-group for p an odd prime denoted by C_{p^k}.

1.2. Discussion of the main result. In order to describe the tower fully, we must determine each stage (P^nX) of the tower, all slices of the tower (P^n_nX), and show that these pieces fit into successive fiber sequences

$$\begin{array}{ccc} P^n_n X & \longrightarrow & P^n X \\ & & \downarrow \\ & & P^{n-1} X \end{array}$$

Our main result in [**Yar15**, Section 4], essentially gives us a blueprint for constructing the slice towers for our selected spectra. We first state the exact form of all nontrivial slices and the dimensions in which they occur. From this information we know the exact dimensions in which the stages of the tower change. We can then determine the form of the spectra that make up each stage and confirm that such spectra do fit into an appropriate tower as the pieces of successive fiber sequences. The following are the key ideas that together form the main result.

Key Idea 1.1. *The nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ are of the form $S^{V_{(a,b)}} \wedge H\underline{B}_{i,j}$.*

$V_{(a,b)}$ is a C_{p^k}-representation obtained from removing a number of irreducible subrepresentations from copies of ρ_G, the regular representation of C_{p^k}. The definition of this representation is given exactly in Definition 3.1. $\underline{B}_{i,j}$ is a Mackey functor obtained by taking quotients of maps from $\mathbb{Z}(i,j)$ to $\mathbb{Z}$ where $\mathbb{Z}(i,j)$ is a slight alteration of the constant Mackey functor. A precise formula is given in Definition 2.3. That such spectra are slices of a particular dimension is shown in the proof of Theorem 3.2.

Key Idea 1.2. *These nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ occur in dimensions $mp^i - 1$ where where $1 \leq i \leq k$ and m is an integer of the same parity as n that occurs in a particular finite range.*

This means that the successive stages of the tower will only change in particular dimensions that are one less that a multiple of a power of p. These dimensions are simply the dimensions of the representations $V_{(a,b)}$ given in Key

Idea 1.1. Additionally, we note that the ranges of i and m are finite and thus the tower itself is finite. This means that we actually have $S^n \wedge H\underline{\mathbb{Z}}$ in the top layer and eventually we get to the trivial spectrum. The remaining stages are briefly described in the next Key Idea.

Key Idea 1.3. *The nontrivial spectra that make up the stages of the tower, that is $P^i S^n \wedge H\underline{\mathbb{Z}}$, are all of the form $S^V \wedge H\underline{\mathbb{Z}}$ where V is a C_{p^k}-representation of dimension n.*

The towers for $S^n \wedge H\underline{\mathbb{Z}}$ are finite and thus at the top of the tower of course we must have $S^n \wedge H\underline{\mathbb{Z}}$ itself. As we work our way down the tower, we see that at level in which the tower changes, a 2-dimensional representation or two trivial representations will be swapped out for another 2-dimensional representation with fewer G-fixed points.

Key Idea 1.4. *The slices are the fibers of maps between successive layers of the spectra described in Key Idea 1.3.*

That is, the slices and stages do in fact fit into fiber sequences. These fiber sequences form the tower beginning at $S^n \wedge H\underline{\mathbb{Z}}$ and terminating at the trivial spectrum. There are patterns that arise in all this data and this shall be discussed more thoroughly in the next section.

1.3. Discussion of the proof. To prove that the data given in the Key Ideas above in fact gives us the slice tower we need to show that the limit is equivalent to $X = S^n \wedge H\underline{\mathbb{Z}}$, the colimit is contractible, every i-dimensional fiber is in fact an i-slice, and these slices and successive stages we've determined do indeed form fiber sequences

$$\begin{array}{ccc} P^i_i X & \longrightarrow & P^i X \\ & & \downarrow \\ & & P^{i-1} X \end{array}$$

As previously mentioned, the towers presented above are finite, so the only interesting work amounts to showing each fiber is an i-slice and that appropriate fiber sequences may be formed.

To show that a spectrum is an i-slice, one must show that it is $\leq i$ and $\geq i$ as defined in [**HHR09**] or [**Hil12**]. As stated in Key Idea 1.1 the candidates for the slices of our tower were of the form $S^{V_{(a,b)}} \wedge H\underline{B}_{i,j}$. In general, to show a spectrum of the form $S^V \wedge H\underline{M}$ is $\geq \dim(V)$, by [**Hil12**, Theorem 3.7], we need only show that $V \subset (m\rho_G - 1)$ for some integer m where $V^G = (m\rho_G - 1)^G$. Additionally, to show $S^V \wedge H\underline{M} \leq \dim(V)$, we can induct on the subgroups of G. By our inductive hypothesis, that is that $S^{|V|} \wedge H\underline{M}$ with a trivial action is $\leq |V|$, and Spanier-Whitehead duality, it will be sufficient to show that $[S^{-\epsilon}, S^{V-t\rho} \wedge H\underline{M}] = 0$ for $tp^k - \epsilon > \dim(V)$ and $\epsilon = 0, 1$. To do so, we really only need to show that the related homology in dimension $-\epsilon$ is trivial and thus are able to make arguments using chain complexes. This method of showing particular spectra are slice is summarized in [**Yar15**, Theorem 5.9].

To show that we have appropriate fiber sequences, we rely heavily on the fact that working p-locally means that our spectra $S^V \wedge H\underline{B}_{i,j}$ and $S^{V'} \wedge H\underline{B}_{i,j}$ will still be equivalent even when V and V' differ by certain subrepresentations of ρ_G. In particular, $H\underline{B}_{i,j} \simeq S^{\lambda_l} \wedge H\underline{B}_{i,j}$ for $l \leq j$ where $\lambda_l : C_{p^k} \to S^1$ is the composition of the inclusion of the p^kth roots of unity with a degree p^ith map on S^1. Essentially, the representation λ_l has too few fixed points to be "seen" by $\underline{B}_{i,j}$ since $\underline{B}_{i,j}$ is trivial on subgroups C_{p^l} for $l \leq j$. Thus, while our descriptions of the slices and layers of the tower may not seem to fit exactly into the correct sequences, they are all equivalent to spectra that do.

References

[Hil12] Michael A. Hill. The equivariant slice filtration: a primer. *Homology, Homotopy, and Applications*. 14(2012), no. 2, 143-166.

[HHR09] Michael A. Hill, Michael J. Hopkins, Douglas C. Ravenel. On the non-existence of elements of Kervaire invariant one. Available as arXiv:0908.3724v4, 2009.

[HHR15] Michael A. Hill, Michael J. Hopkins, Douglas C. Ravenel. The slice spectral sequence for RO(C_{p^n})-graded suspensions of $H\underline{\mathbb{Z}}$ I. In preparation.

[Yar15] Carolyn Yarnall. The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic p-groups. Available as arXiv:1510.02077.

Department of Mathematics, University of Kentucky, Lexington, KY 40506

E-mail address: carolyn.yarnall@gmail.com