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# A user's guide: The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic $p$-groups

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## 2. Metaphors and imagery

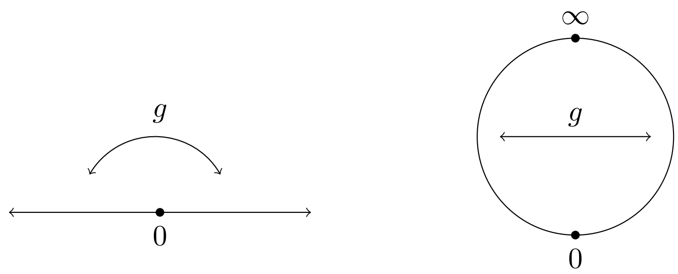
**2.1. Equivariant stable homotopy theory.** Before discussing the imagery I use concerning the specific results of the paper [**Yar15**] or even the slice filtration itself, it will be best to briefly describe how I think about the basic objects used, $G$-spectra and Mackey functors.

2.1.1. *$G$-spaces and representation spheres.* As the story of $G$-spectra is most fully told by relying on the collection of $G$-spaces called representation spheres, it is most fitting to begin here. A $G$-space is a topological space with an action of a group $G$. What we imagine is a collection of points that become permuted in some way when applying elements from the group. The action induced by any given element may be *trivial* in which no points are moved or it may be *free* in which no points are fixed. An important collection of points in the equivariant context is the collection of fixed points for a given subgroup $H$ of $G$. In reality, we actually need to keep track of the fixed points for all subgroups $H$ of $G$ and thus, one consideration we always carry along with us is the structure of the group lattice.

One type of $G$-space that is integral to the equivariant stable field is a representation sphere $S^V$. What we mean by this notation is the one point compactification of the space associated to the representation $V$. Such a vector space may be thought of as a $G$-space with action induced in the obvious fashion. That is, each element of $G$ acts on the space as the associated element from the general linear group. Thus, we imagine taking a space $\mathbb{R}^n$ with permuted points and gathering everything at infinity to one point while carrying that group action along. Forming the sphere in this way forces the points at 0 and $\infty$ to always be fixed. The following examples are common ones that I like to keep in mind.

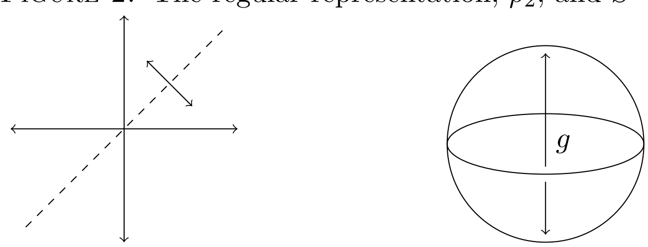
Example 2.1. Consider the 1-dimensional sign representation $\sigma : C_2 \to GL(\mathbb{R})$. The associated representation space is depicted on the left in Figure 1. When $\mathbb{R}$ is thought of as a $C_2$-space with an action induced by $\sigma$, 0 is always fixed by the $C_2$-action and the action of the nontrivial element $g$ swaps the positive and negative points. Its associated representation sphere is depicted on the right.

FIGURE 1. The sign representation, $\sigma$, and $S^{\sigma}$




EXAMPLE 2.2. Then the regular representation of $C_2$ has a decomposition $\rho_2 = 1 + \sigma$. $\rho_2$ is then a 2-dimensional representation with its associated space shown in Figure 2 on the left. As a $C_2$-space, the action of the trivial element flips the $x$- and $y$-coordinates, fixing the diagonal. Its associated representation sphere is depicted on the right. We can see then that the equator (coming from the diagonal) is always fixed by the action of $C_2$ and the upper and lower hemispheres are swapped when acted on by the nontrivial element from $C_2$.

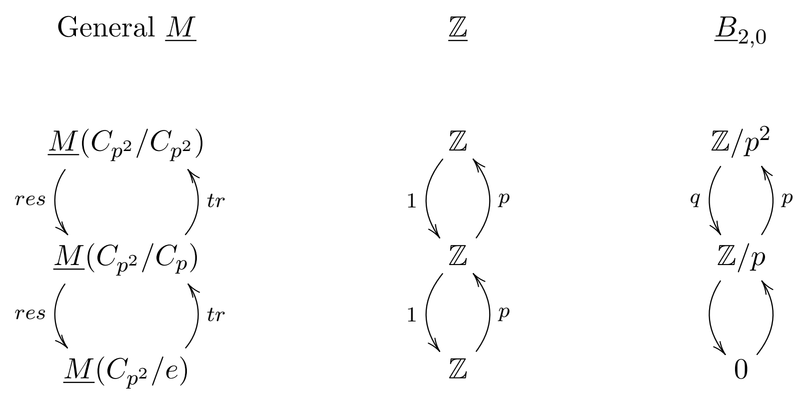
FIGURE 2. The regular representation, $\rho_2$, and $S^{\rho_2}$




2.1.2. *G-spectra.* In the context of stable homotopy theory, one may think of a spectrum as a sequence of spaces arranged in an ordered infinite line with structure maps running between them. When considering $G$-spectra, we must first imagine that each space used in forming the spectrum is a $G$-space. To picture "naive" $G$-spectra one need only imagine a similarly infinite line of $G$-spaces. However, "genuine" $G$-spectra, those considered in the paper, are a bit more complicated. The $G$-spaces that fit together to form a spectrum are instead indexed on repesentations of $G$. There are still infinitely many but they are not arranged linearly. Rather, we could think of the $G$-spaces arranged in a sort of directed lattice whose structure is determined by the collection of $G$-representations we are indexing on. Two $G$-spaces, say $X_V$ and $X_W$, are connected in the lattice if $V \subset W$. Thus, not only do the spaces that form the spectrum have an action of $G$ but we must also remember that the way they are arranged carries information from the group $G$ as well.

2.1.3. *Mackey functors.* When computing the homotopy "groups" of a $G$-spectrum, we actually get a collection of groups that fit together to form an object called a *Mackey functor*. A Mackey functor, $\underline{M}$, can be thought of as a bifuntor from the orbits of $G$ to abelian groups. It is also useful to think of a Mackey functor as a lattice but this time, the lattice structure is determined by the group lattice of $G$. Additionally, as it is a bifunctor, each edge in the lattice has two directions. In [**Yar15**], we only consider $G$ to be a cyclic $p$-group, and thus all $G$-Mackey functors in this context can be thought of as "ladders"; the groups $\underline{M}(G/H)$ are the rungs and the contravariant (restriction) and covariant (transfer) morphisms form the sides. See Figure 3 for examples of Mackey functor diagrams. Note that $\underline{\mathbb{Z}}$ is the constant Mackey functor and $\underline{B}_{2,0}$ is a particular case of a Mackey functor defined in [**HHR15**] and used extensively in [**Yar15**].

FIGURE 3. Examples of $C_2$-Mackey functors




Picturing a Mackey functor in this way is especially useful for performing computations such as homology computations with chain complexes of Mackey functors. In such work, one can then picture a sort of commutative diagram of lattices. Additionally, this imagery impresses upon us the important role that subgroups play in the equivariant realm as each subgroup corresponds to a different point in the lattice or "rung" on the ladder.

**2.2. The slice filtration.** Most often, the way I imagine the slice filtration is set against the backdrop of the Postnikov tower. The Postnikov tower builds a spectrum one homotopy group at a time and thus one can imagine the homotopy groups as neatly stacked blocks. On the other hand, the slice tower conjures an image of smearing out the homotopy groups of a $G$-spectrum.

Below, we see the homotopy groups of $P_nX$ and $P_{n-1}X$ in the Postnikov tower:

| Dimension | $P_nX$ | $P_{n-1}X$ |
|---|---|---|
| | $\cdots$ | $\cdots$ |
| $n+2$ | 0 | 0 |
| $n+1$ | 0 | 0 |
| $n$ | $\pi_n(X)$ | 0 |
| $n-1$ | $\pi_{n-1}(X)$ | $\pi_{n-1}(X)$ |
| $n-2$ | $\pi_{n-2}(X)$ | $\pi_{n-2}(X)$ |
| $\cdots$ | $\cdots$ | $\cdots$ |

From this, it is easy to see that the fiber of the map $P_nX \to P_{n-1}X$ has its only nontrivial homotopy group in dimension $n$, namely, $\pi_n(X)$.

If we consider the layers of the slice tower $P_nX$ in terms of stacked integer-graded homotopy groups, we do not get such a straightforward picture. This is because each homotopy group in dimension $n$ and below need not be the same as the homotopy groups of $X$. For example, consider $P_0X$ and $P_{-1}X$ shown below:

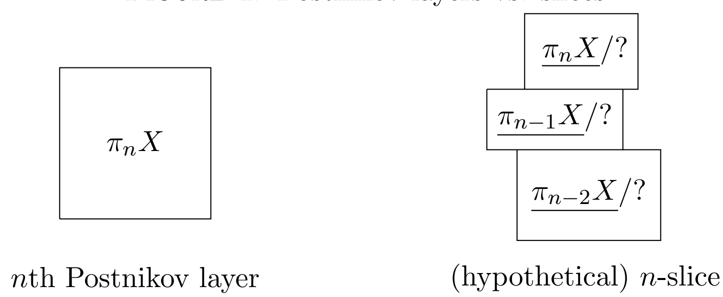
| Dimension | $P_0X$ | $P_{-1}X$ |
|---|---|---|
| | $\cdots$ | $\cdots$ |
| 1 | 0 | 0 |
| 0 | $\underline{\pi_0(X)}/*$ | 0 |
| $-1$ | $\underline{\pi_{-1}(X)}$ | $\underline{\pi_{-1}(X)}$ |
| $\cdots$ | $\cdots$ | $\cdots$ |

The Mackey functor $\underline{\pi_0(P_0X)}$ is really a quotient of $\underline{\pi_0(X)}$. This is because some of the slice cells we kill maps from in this instance are not as connected as their underlying sphere. In particular, $S^\sigma$ is not 0-connected even though the underlying space, $S^1$ is.

We pause to note that there is actually a nice description of 0-slices due to Hill in [**Hil12**] and they are in fact Eilenberg-MacLane spectra. However, for larger slices we do not have a strong understanding. We do not know the homotopy groups in stages above the zeroth in general. In fact, the layers are often so jumbled that the fibers need not be Eilenberg-MacLane spectra and moreover, are often difficult to determine even in individual cases.

We can still think of moving up the slice tower as building the $G$-spectrum using homotopy information, it's just that we are not adding one integer graded homotopy group at a time. In constructing the $n$th stage of the slice tower, homotopy groups in dimension $n$ and below may be altered. Thus, we often see $G$-spectra with many nontrivial homotopy groups as slices. So while one might imagine the "building blocks" in the context of the Postnikov tower as uniform cubes, the building blocks when viewed through the lens of the slice tower can be objects of a variety of sizes and shapes. Figure 4 gives a general visual idea of the comparison between a single Postnikov building block and a single slice building block.

FIGURE 4. Postnikov layers vs. slices




It is important to keep in mind that what is depicted in Figure 4 is only a vague possibility of what we might think about an $n$-slice. While it is true that we won't ever see any part of an $n$-slice building block above dimension $n$, we don't know in general how many lower dimensions the block might touch. We also don't really know the exact "shape" only that it makes sense to think of parts of the block as "smaller" than the homotopy "cubes" since they arise from quotients of the Postnikov layers.

**2.3. Concerning main result.** Now, how should one think about the specific slice towers for $C_{p^k}$-spectra of the form $S^n \wedge H\underline{\mathbb{Z}}$? While we might often describe building the tower from the bottom up, I find it useful to imagine beginning at the top and working down when trying to picturing exactly what the slice filtration does to the given spectrum. In the particular case considered in [**Yar15**], every nontrivial sequence in the tower looks like

$$\begin{array}{ccc} S^V \wedge H\underline{M} & \longrightarrow & S^W \wedge H\underline{\mathbb{Z}} \\ & & \downarrow \\ & & S^{W'} \wedge H\underline{\mathbb{Z}} \end{array}$$

where $\underline{M}$ is some $C_{p^k}$-Mackey functor and $V, W$, and $W'$ are $C_{p^k}$-representations, $W$ and $W'$ having dimension $n$. In this depiction, we mean that $P_n X \simeq S^W \wedge H\underline{\mathbb{Z}}$, $P_{n-1}X \simeq S^{W'} \wedge H\underline{\mathbb{Z}}$, and the fiber of the map, or slice, is $P_n^n X \simeq S^V \wedge H\underline{M}$. Now the question is, what exactly does this mean?

This is the way I "read" such a tower: the slice $S^V \wedge H\underline{M}$ encodes the information that tells us how the stages of the tower are changing. In particular, it is the Mackey functor $\underline{M}$ that determines this; the representation $V$ is essentially forced by how the tower is changing. Furthermore, as mentioned above in Topic 1, these representations can be written in many equivalent ways so it doesn't really make sense to think of $V$ as static. Each Mackey functor $\underline{M}$ is of a certain type (really $\underline{M} = \underline{B}_{i,j}$, see Figure 3 for an example) and I think of them being color coded to tell us how $W$ changes into $W'$. Each different color prescribes a particular change in the tower. Consider the following example:

Example 2.3. Here is a (modified) portion of the slice tower for $S^{10} \wedge H\underline{\mathbb{Z}}$ with $G = C_9$:

$$\begin{array}{ccc}
S^V \wedge H\underline{green} & \longrightarrow & S^{4+2\lambda_1+\lambda} \wedge H\underline{\mathbb{Z}} \\
 & & \downarrow \\
S^{V'} \wedge H\underline{red} & \longrightarrow & S^{2+2\lambda_1+2\lambda} \wedge H\underline{\mathbb{Z}} \\
 & & \downarrow \\
 & & S^{2+\lambda_1+3\lambda} \wedge H\underline{\mathbb{Z}}
\end{array}$$

The "green" Mackey functor invokes a change of two trivial representations for one $\lambda$ while the "red" Mackey functor swaps one $\lambda_1$ for one $\lambda$. Every $\lambda$ representation is 2-dimensional so the underlying dimension is preserved. The order in which we write the subrepresentations (trivial, $\lambda_1$, $\lambda$) lists them from the most fixed point to least fixed points under the group action.

More generally, the change from $W$ to $W'$ is always a 2-dimensional subrepresentation of the regular representation for $C_{p^k}$ being swapped out for another 2-dimensional subrepresentation that has fewer fixed points. When looking at each tower from top to bottom, they seem to swap out as many representations as possible with a large number of fixed points first before swapping out representations with fewer fixed points. In particular, each one of the towers begins with a sequence of changing trivial representations to $\lambda$-type representations. Naturally, one would like to know why this is the case. In order to better understand this, one must look instead at the pattern of Mackey functor types as these determine the changes in representations.

The various "colors" of $\underline{M}$ (or types of $\underline{B}_{i,j}$) follow a pattern that relies on the $p$-adic valuation of $n$ for each $(n-1)$-slice. Figures 5 and 6 give the order the Mackey functor types for the nontrivial slices in the given towers for $G = C_9$. At the top of each we see the same type of functors until the power of $p = 3$ decreases. Then we can see that anytime the slice dimension $n-1$ has a larger $p$-adic valuation for $n$ we have a different type of functor. Why? Consider, for example, the slice in dimension $12(3) - 1$. We could also write this as $4(3)^2 - 1$

and thus might expect that the functor type is more similar to those slices in dimensions $m(p)^2 - 1$ at the top. This is indeed the case and furthermore is the reason why I think of these as green (closer to blue) rather than red.

FIGURE 5. $S^7 \wedge H\underline{\mathbb{Z}}$

| Slice Dimension | Mackey functor type |
|---|---|
| $5(3)^2 - 1$ | blue |
| $3(3)^2 - 1$ | blue |
| $5(3) - 1$ | red |
| $3(3) - 1$ | green |

FIGURE 6. $S^{16} \wedge H\underline{\mathbb{Z}}$

| Slice Dimension | Mackey functor type |
|---|---|
| $14(3)^2 - 1$ | blue |
| $12(3)^2 - 1$ | blue |
| $10(3)^2 - 1$ | blue |
| $8(3)^2 - 1$ | blue |
| $6(3)^2 - 1$ | blue |
| $14(3) - 1$ | red |
| $12(3) - 1$ | green |
| $10(3) - 1$ | red |
| $8(3) - 1$ | red |
| $6(3) - 1$ | green |

Another property to note is that in each of the towers depicted the red/green pattern that follows the blue slices is the same. In particular, one red slice appears first and each tower ends in green. This is due to the fact that the difference between the suspensions is 9, exactly the size of the group we are considering. We could guess (and we'd be right!) that the tower for $S^{25} \wedge H\underline{\mathbb{Z}}$ would also begin with a number of blue slices, then one red, then a longer $p$-adic pattern of greens and reds, ending with green. The reason behind this property is intrinsically tied to the interplay between the slice tower and suspensions by regular representation spheres; the dimension of such spheres is of course the order of $G$. The exact relationship between the slice tower and suspensions is discussed in Topic 3 below.

One last note regarding these patterns is that the number of color block types corresponds to the power of $p$ in the group $G$. When $G = C_p$, we see only one type of Mackey functor. When $G = C_{p^2}$ as in the tables above, we see two: blue and red/green. When $G = C_{p^3}$ we see three (e.g. blue, red/green, purple/turquoise/white). Additionally, the size of the blocks in each tower is the same (except in the outlier case where the $n$ in $S^n \wedge H\underline{\mathbb{Z}}$ is a multiple of $p$). This property is a result of the relationship between the slice tower and the notion of restricting to subgroups in the equivariant setting. This relationship is further discussed in Topic 3.

## References

[Hil12] Michael A. Hill. The equivariant slice filtration: a primer. *Homology, Homotopy, and Applications*. 14(2012), no. 2, 143-166.

[HHR15] Michael A. Hill, Michael J. Hopkins, Douglas C. Ravenel. The slice spectral sequence for $\mathrm{RO}(C_{p^n})$-graded suspensions of $H\underline{\mathbb{Z}}$ I. In preparation.

[Yar15] Carolyn Yarnall. The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic $p$-groups. Available as arXiv:1510.02077.

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