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A user's guide: The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic p-groups

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4. Colloquial summary

Just as a doctor may use a CAT scan to better understand what is happening in the human body, a mathematician often uses a given tool to better understand the structure and properties of certain mathematical objects. In addition to using the tool to study an object, one might, conversely, study the tool itself in order to better understand what it actually does and what information it gives us about the objects it is applied to.

For example, it makes sense to study what a CAT machine is and how it works because then we know what it will see in the human body, and what it will miss. The purpose of the paper [**Yar15**] is a little of both: using a given tool to study particular objects in an attempt to study the tool itself. In the paper itself, we investigate what our tool, the *slice filtration*, does to a particular family of objects called *G-spectra*. In conjunction with other work [**HHR15**] we hope that our answers might tell us more about the slice filtration itself.

4.1. The objects. The objects we consider, G-spectra, are generalizations of objects called G-spaces. As the latter are a bit more tangible, we now focus on such objects rather than G-spectra. How should one think of a G-space? First, one might think about a space as a collection of points but most often these points are all connected to form a continuous shape. For example, spaces we often consider in this context are circles, spheres, or higher-dimensional versions of spheres. Then what does the "G" tell us? G represents another mathematical object called a group. The most straightforward example of a group is a collection of numbers with an operation, like addition. A G-space is a collection of points that get jumbled around in a way that is dictated by the types of elements (or numbers) in the group G.

There is an important restriction on how the points are being jumbled: they can only be swapped around in place of other points but the overall shape of the space cannot change. For example, we might consider a circle and the way the points will be rearranged is by rotating the circle by a given amount. This type of restriction means that by using G to swap around points, we are really capturing data about the symmetry of the space.

Additionally, we can plug many different groups in for G. G could be the simplest type of group, a group containing only one number. In this case, there is actually no difference between a G-space and an ordinary space. The larger and more complicated G becomes, the more difficult it will be to study any corresponding G-space. Our goal in this setting is to study not only the shape of the object but also how the points are being rearranged.

The G-spectra that we analyze in [**Yar15**] are given the name "$S^n \wedge H\underline{\mathbb{Z}}$" where n can be any positive whole number. The part $H\underline{\mathbb{Z}}$ is an object that encodes a certain type of information. The $\underline{\mathbb{Z}}$ tells us what this type of information is being represented. The "S^n" are n-dimensional spheres and "$\wedge$" is a type of product. Essentially, the "$S^n \wedge$" part tells us to shift the object $H\underline{\mathbb{Z}}$ to dimension n. Notably, when we shift the object, we are doing it in a way that maintains its "type". It turns out that $S^n \wedge H\underline{\mathbb{Z}}$ is considered to be a relatively simple object by initial observation but our tool will allow us to see what is really happening beneath the surface, so to speak.

4.2. The tool. A common theme in many areas of study is learning about the properties of an object by seeing how the object is constructed out of smaller pieces. A microbiologist may learn about an organism by studying its cells or a chemist may study a substance by determining what elements it is comprised of.

Similarly, mathematicians will often break down objects into smaller pieces and gain insight by studying how the object is built out of the smaller pieces. The slice filtration is a tool that helps to analyze how a G-spectrum is built out of "smaller" G-spectra. This ends up being a bit difficult as it is not immediately apparent what in fact we mean by "smaller" or "simpler" G-spectra.

One way to think about the slice filtration is that it provides building blocks and a blueprint for assembling the blocks into the spectrum we are analyzing. However, the blocks it uses are a bit mysterious. If we remove the "G" and think only about spectra, we would see that a spectrum can be built as a sort of stack of uniform type blocks. Every layer is even, like building a wall out of Legos.

For a G-spectrum, our wall looks more like a game of tetris. The layers are not even; each block may have a different shape and be a part of many layers. Difficulties arise because in general we don't even know exactly what all the blocks look like! In the paper [**Yar15**] we determine the building blocks and how they are assembled to form the objects $S^n \wedge H\underline{\mathbb{Z}}$. The fact that these superficially simple objects have fairly complicated structures as determined by the slice filtration demonstrates how rich G-spectra really are.

References

[HHR15] Michael A. Hill, Michael J. Hopkins, Douglas C. Ravenel. The slice spectral sequence for RO(C_{p^n})-graded suspensions of $H\underline{\mathbb{Z}}$ I. In preparation.

[Yar15] Carolyn Yarnall. The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic p-groups. Available as arXiv:1510.02077.

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