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A user's guide: Dynamics and fluctuations of cellular cycles on CW complexes

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1. Key insights and central organizing principles

In [**CCK**], we consider the stochastic motion of subcomplexes in a CW complex, and explore the implications on the underlying space. This is a direct generalization of a random walk on a graph, and allows a higher-dimensional notion of electrical current, known as extended empirical or stochastic current, to be defined. Empirical currents provide a unique link between the random process and the topology of the complex. The ideas underlying empirical currents are a marriage of Langevin dynamics on smooth manifolds and discrete dynamics of random walks on graphs. In the manifold setting, the random process we consider is easier to express and the ideas behind empirical currents are more intuitive. This is in contrast to the discrete setting, where the precise formula for average current is simpler to state. In this user guide, we will focus on both cases and play them off one another to gain better intuition for the random processes under study, as well as their topological implications.

The first rigorous topological study of empirical current was for a particle taking a random walk on a graph in [**CKS13**]. For a particle undergoing stochastic motion on an arbitrary smooth manifold, current was first considered in [**CCMT09**]. However, both of these studies were for the motion of 0-dimensional objects. The focus of [**CCK**] is to analyze the motion of cycles of arbitrary dimension on a CW complex.

1.1. Random walks on graphs. Let X be a connected graph, by which we mean a connected, one-dimensional CW complex. Consider a particle taking a random walk on X. The particle sits on the 0-cells, or vertices of X, and after waiting some random time, hops across a 1-cell, or edge, to an adjacent vertex. This defines a Markov process with state space given by the set of vertices, and transitions between states occur through the edges. The rate at which the particle moves is determined by external data. The typical set-up is to equip each vertex i and each edge α with real numbers E_i and W_α. If α is an edge connecting vertices i and j, then the rate at which the particle will hop from i to j across α is given by

$$H_{ji}(\beta) = e^{-\beta(W_\alpha - E_i)}\,, \tag{1}$$

where β is a fixed positive real number[1]. We assemble these rates into a matrix $H = [H_{ij}]$ acting on $C_0(X;\mathbb{R}) = Z_0(X;\mathbb{R})$. The probability distribution of a particle taking a random walk on X starting at vertex i is governed by the Kolmogorov, or Fokker-Planck, equation

$$\frac{dp}{dt} = Hp \qquad p(0) = i\,, \tag{2}$$

for $p = p(t) \in C_0(X;\mathbb{R})$. When $p(t)$ is normalized, so that $\sum_{i\in X_0} p_i(t) = 1$, Eq. (2) governs the flow of probability on the graph.

It should be stressed that the homology class doesn't change throughout the random walk. In fact, one can (and should) think of 'hopping' from i to j across α as adding $\partial\alpha$ to i: $j = i + \partial\alpha$. This idea of hopping being the same as adding boundaries of cells one dimension higher is crucial for the more general situation to follow. The time which we let the particle evolve is known as the *evolution time* and is denoted by τ. We are interested in an explicit formula for the particle's trajectory under various limits on the Markov process.

Key Idea 1.1. *The* average current *is given by the real homology class of the particle's trajectory on X, divided by the evolution time.*

It is easiest to see why this homology class deserves to be called current in the case of a manifold, which we now discuss.

1.2. Langevin dynamics. The Kolmogorov equation given above governs distributions of trajectories rather than describing a particle's actual trajectory. Let us spell out this stochastic motion in detail for manifolds.

Fix a smooth, compact, Riemannian manifold (M, g), together with a Morse function $f : M \to \mathbb{R}$. We also equip M with a *stochastic vector field* ξ. By this, we mean a time-dependent vector field of M satisfying Gaussian, Markovian statistics. This means that $\xi(t)$ is not only a vector field on M, but is also a Gaussian random variable in the sense of probability theory. Its first two moments are

$$\langle \xi(x,t)\rangle = 0\,, \quad \langle \xi(x,t)\xi(x,t')\rangle = \beta^{-1}\delta(t-t')g(x)\,, \tag{3}$$

where g is the Riemannian metric. The delta-correlation in time reflects the Markovian property, and since the variable is Gaussian, the first two moments determine all the higher moments. A particle on M will then move according to the Langevin equation

$$\frac{dx}{dt} = u(x,t) + \xi(x,t), \tag{4}$$

[1]It can be shown that the rates for such a process can always be written in this form.

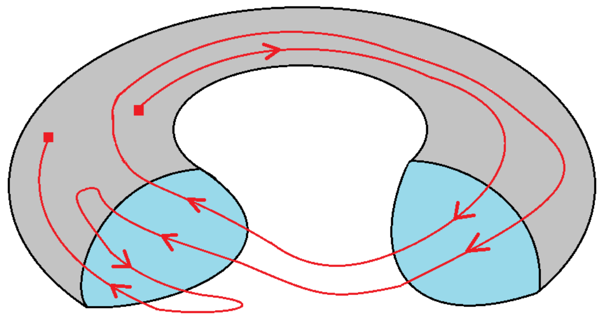

FIGURE 1. Traditional electric currents as considered in [**CCMT09**]. The stochastic trajectory of an electron is shown in red, and two cross-sections are shown in blue. The number of intersections is different, but the signed intersection indices agree.

where locally, $u(x,t) = -\nabla f(x,t)$ is given by the gradient flow. The main point is that the stochastic motion is controlled by a deterministic piece and a stochastic piece.

A solution to Eq. (4) is a stochastic trajectory, and is represented by a path $\eta : [0,\tau] \to M$. For long times τ, one can assume the path is closed [**CCMT09**, p. 6], and so the trajectory can be represented by $\eta : S^1 \to M$. This gives rise to a class in the real-bordism homology of M

$$Q_{\tau,\beta}(u) = \tfrac{1}{\tau}[\eta] \in H_1(M;\mathbb{R}),$$

known as the *average empirical current density associated to* η *and duration* τ.

The prototypical example from which this terminology is motivated is that of an electron in an electrical wire $M = S^1 \times D^2$, as in Figure 1. Connecting the wire to a battery gives rise to the deterministic term u, and random collisions with other particles and impurities in the wire give rise to the stochastic vector field ξ. For an oriented cross-section $\alpha : \{p\} \times D^2 \to S^1 \times D^2$, the *current at* α is the number of charged particle crossings at α per unit time. This is the (signed) number of crossings of the electron through α, and is given by the intersection pairing $[\eta] \cdot [\alpha] = N_+ - N_-$, where $N_\pm$ is the number of crossings with positive/negative orientation. In this same sense,

$$\tfrac{1}{\tau}[\eta] \cdot [\alpha] \in \mathbb{R}$$

is the *average current at* α associated to the electron. The actual average current at α is a sum over all electrons.

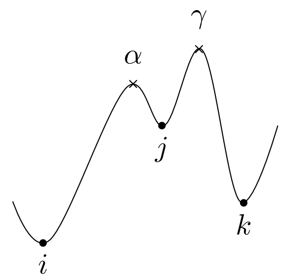

FIGURE 2. The height function on a one-dimensional manifold. The local minima, marked with dots, give rise to 0-cells and the local maxima, marked with crosses, yield the 1-cells.

If the manifold is one-dimensional, then the Morse function f determines a graph structure on M[2], as shown in Figure 2. The vertices are given by the local minima $\{i, j, k\}$ and the edges by the local maxima $\{\alpha, \gamma\}$. Initially, a particle on M will fall into a local minimum under the gradient flow; suppose it is i. If the vector field ξ becomes large enough (which is rare since it is mean zero), it can push the particle out of the minimum and up towards a maximum. Suppose that ξ is so large that the particle crosses the maximum at α. It will then fall down into the minimum at j. This process of ξ moving the particle from one minimum to another is identical to the particle jumping across an edge on a graph (and is precisely what is happening on the associated CW complex). The rate at which a hop occurs is governed by Eq. (1), with $E_i := f(i)$ and $W_\alpha := f(\alpha)$. In general, the larger the difference $f(\alpha) - f(i)$, i.e. the 'higher' the barrier, the less likely the particle is to cross α. It is interesting to note that the rate at which a jump from i to j occurs does not depend on j, as seen in the rate of Eq. (1), as well as the intuitive picture of Figure 2. Furthermore, this picture makes it clear that the absolute barrier height $f(\alpha)$ does not determine the rates, but rather the relative differences $f(\alpha) - f(i)$.

1.3. Extended empirical currents. The additional complexity of stochastic motion in higher dimensions is not due to the ambient space in which the random motion occurs. Rather, it is because of the moving objects themselves. Unlike [**CKS12**] and [**CKS13**] which studies the motion of 0-dimensional objects, [**CCK**] studies the stochastic motion of objects of arbitrary dimension (hence the word 'extended'). With the intuition of particles moving on graphs and manifolds, we now describe the main stochastic process on CW complexes studied in [**CCK**].

Fix a finite, connected CW complex X of dimension d, and an initial cycle $\hat{x} \in Z_{d-1}(X; \mathbb{Z})$. This cycle should be thought of as an initial condition, analogous to the starting vertex of a random walk on a graph. The initial cycle can 'hop'

[2]In general, a Morse function gives rise to a CW decomposition as follows. The open cells are indexed by the local extrema of f, and are explicitly given by the unstable manifolds: the points which flow out of an extremum under the (negative) gradient flow of f.

across a d-cell to form a different $(d-1)$-cycle, given by adding the boundary of that d-cell. This is just as how a particle hops across an edge on a graph. By prescribing real numbers E_i and W_α to each $(d-1)$ cell i and d-cell α, we can form rates which govern how the cycle will move on X, analogous to Eq. (1). Rigorously, the state space of this Markov process in higher dimensions consists of the set of real $(d-1)$-cycles homologous to $\hat{x}$. An elementary transition from state z to z' occurs if the cycles differ by adding or subtracting a single boundary of a d-cell. Generic transitions are compositions of elementary ones. The *average empirical current associated to $\hat{x}$ is*

$$Q_{\tau,\beta}(E,W) = \tfrac{1}{\tau}[x_\tau] \in H_d(X;\mathbb{R})$$

where $[x_\tau]$ is the homology class of the cycle after evolving for time τ. We are interested in explicit formulas for this homology class.

The intuition behind the process and formula for current described above arise from the smooth picture. To play the two situations off each other, suppose the CW complex arose from a Morse decomposition on a smooth, Riemannian manifold M. Take the initial cycle to be a closed submanifold $\eta_0 : N \to M$. Initially, η_0 will evolve deterministically according to $-\nabla f$, and will tend to the $(d-1)$-skeleton of M as determined by f (this is the definition of the CW structure). The stochastic vector field ξ will allow the cycle to fluctuate in some small neighborhood of the $(d-1)$-skeleton of M ('small' since ξ is mean-zero). On longer time scales, ξ can push a segment of the evolved cycle η_t and move it off the $(d-1)$-skeleton, into a d-cell. If the random field ξ is large enough, this segment of η_t can move up to a critical point of index d, against the gradient flow, and cross the critical point. This is completely analogous to the particle on a 1-dimensional manifold getting pushed up towards, and eventually crossing the local maxima. For either a particle or extended object, once it crosses the critical point, the gradient flow will push it back down into the $(d-1)$-skeleton; nothing special is needed of the noise for the object to 'relax', or literally flow, back into the $(d-1)$-skeleton. Throughout this process, the homology class never changes, since we have only added the boundaries of cells to it during its evolution. The *average empirical current associated to* η_0 is

$$Q_\tau(f) = \tfrac{1}{\tau}[\eta_\tau] \in H_d(M;\mathbb{R}),$$

where τ is the duration for which we let the cycle evolve and we use real bordism homology.

1.4. A formula for average current. The stochastic dynamics described above, both smooth and discrete, as well as the average current, are only interesting if the Morse function f, or the numbers $\{E_i, W_\alpha\}$ vary in time. Indeed, if the rates are fixed, an object will tend to the configuration of minimal energy (value of E or f) and with overwhelming probability, remain there, resulting in zero current.

Key Idea 1.2. *Current generation can only occur if the parameters vary in time.*

The situation is more interesting if we drive the system by varying the numbers E and W periodically in time. The space of parameters $\mathcal{M}_X$ parameterizes the possible rates for the Markov process. We are interested in the limit of observing many periods of γ, and implicitly take $\tau = N\tau_D$, where N is a large integer and τ_D is the period of γ.

Practical formulas for the average current can be obtained after employing two limits. The first is the low-temperature limit, in which $\beta \to \infty$. This is also known as the low-noise limit, since the dynamics become more and more deterministic as β grows (see Eq. (3)). The other limit is the *adiabatic* limit, in which $\tau_D \to \infty$. The term adiabatic appreciates the fact that the parameters change extremely slowly in time.

For τ_D sufficiently large, the average current can be written explicitly in terms of 2 ingredients. The first is Kirchhoff's higher dimensional network theorem of [**CCK15a**], which gives the unique solution to Kirchhoff's laws in an electrical network. The second appears in [**CCK15b**], and is a higher dimensional notion of the Boltzmann distribution. We postpone their discussion to the next section. For now, we state the main result of [**CCK**]:

THEOREM 1.3 ([**CCK**]). *Let X be a finite, connected CW complex of dimension d. For a sufficiently generic γ,*

$$\lim_{\tau_D \to \infty} Q_{\tau_D,\beta}(\gamma) = \int_0^1 K(\dot{\rho}^B) dt\,,$$

where K is the Kirchhoff operator and ρ^B is the Boltzmann distribution. Furthermore, in the low-temperature, adiabatic limit, the average current quantizes:

$$\lim_{\beta \to \infty} \lim_{\tau_D \to \infty} Q_{\tau_D,\beta}(\gamma) \in H_1(X; \mathbb{Z}[\tfrac{1}{D}]) \subset H_1(X; \mathbb{R})$$

where D is determined by combinatorial data of X.

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