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A user's guide: Dynamics and fluctuations of cellular cycles on CW complexes

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2. Metaphors and imagery

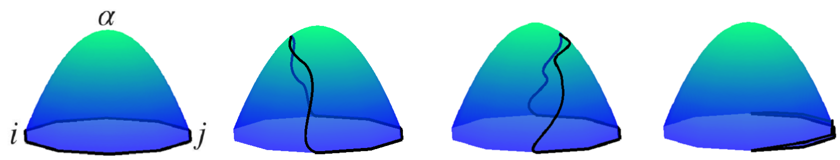
In this section, we visualize the concepts described in the previous section. We first give imagery concerning the stochastic process on general CW complexes, together with their smooth generalizations. We then discuss the two key components to the average current: Kirchhoff's network theorem/solution and the Boltzmann distribution. We discuss the inherent geometric nature of these two pieces and show how they can be used to think about the average current.

2.1. Visualizing the stochastic process. A simple picture of a transition in higher dimensions is displayed in Figure 1 for a 2-dimensional CW complex, although the general picture is very similar. Recall that to interpolate between the smooth and discrete cases, we describe a CW complex arising from a Morse decomposition. An elementary transition, as described in the previous topic, is shown in Figures 1a-1d. The cycle $\hat{x}_0 = i + j$ shown evolves in the manifold by jumping 'off' i and 'across' α to

$$\hat{x}_1 = \hat{x}_0 + \partial\alpha = \hat{x}_0 + j - i = 2j\,.$$

This is one aspect of working in higher dimensions which is significantly different from the graph case. An elementary transition on a graph only requires an edge, or 1-cell. On a CW complex of arbitrary dimension d, we must specify a d-cell to hop across and a $(d-1)$-cell to hop 'off'. It is important to note that only the first and last figures take place on the CW complex, whereas the intermediate transition lies in the smooth manifold. We only use the smooth picture for motivation, so we think of this transition occurring instantaneously on the CW complex.

If α had more boundary components, the situation would be more complex. For example, take $\partial\alpha = m_1 + \cdots + m_k$ with $\hat{x}_0 = \sum_j n_j m_j$ for some $(d-1)$-cells m_j and integers n_j. In the above scenario, in which $\hat{x}_0$ moves off m_i along α, the



(A) (B) (C) (D)

FIGURE 1. An elementary transition on a CW complex. A 2-cell α is shown, given by the local maximum; the left side of its boundary is a 1-cell i and the right is the 1-cell j. The transition occurs from the i to j along α.

evolved cycle would be

$$\hat{x}_1 = \hat{x}_0 - n_i \partial \alpha = \sum_{j \neq i} (n_j - n_i) m_j ,$$

so that $\hat{x}_1$ has no incidence with m_i, and the incidence x_0 has with i has been subtracted from the rest of the incidences.

We now turn our attention to the visualization of the current and the pieces it is constructed from: the Kirchhoff solution and the Boltzmann distribution.

2.2. Spanning trees and co-trees. Kirchhoff constructed a solution to the network problem on graphs (see [**CKS13**] and [**CCK15a**]) using spanning trees. A spanning tree in higher dimensions can be thought of as an appropriate truncation or approximation to the CW complex X. The point is that spanning trees do not have any d-cycles, but still contain enough of X to have all the rational homology of X in lower degrees. From the viewpoint of current, they satisfy one crucial property: let $b \in B_{d-1}(X; \mathbb{Q})$ be any boundary of X and let T be any spanning tree. There is a unique d-chain $K_b^T \in C_d(T; \mathbb{Q})$ such that[1]

$$-\partial_T K_b^T = -\partial K_b^T = b.$$

This generalizes the fact that on a graph, every spanning tree contains a unique path between any two vertices (their difference being a boundary). The solution to the network problem is given by taking a weighted sum of such operators

$$K = \tfrac{1}{\Delta} \sum_T w_T K^T : B_{d-1}(X; \mathbb{Q}) \to C_d(X; \mathbb{Q}).$$

The other ingredient is the Boltzmann distribution. Originally defined as an energy distribution for particles in a gas, it was shown in [**CCK15b**] that the Boltzmann distribution can also be used to describe harmonic forms on CW complexes. The distribution is written as a sum over certain subcomplexes known as spanning co-trees. These subcomplexes are again approximations to X, and contain enough of X to reproduce its rational homology in degree $(d-1)$. They

[1]The minus sign on ∂ is physically motivated, and does not affect the relevant algebra. The topologist can omit it.

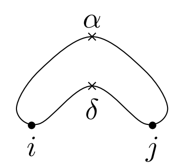


FIGURE 2. The height function on S^1, giving rise to a CW decomposition, with parameters shown as in segment (ii) of the driving protocol.

are so useful because they satisfy the following. Let $[x] \in H_{d-1}(X;\mathbb{Q})$ be any homology class of X and let L be any spanning co-tree. Then there is a unique cycle $\psi_L([x]) \in Z_{d-1}(L;\mathbb{Q})$ representing $[x]$. Very roughly, a spanning co-tree should be thought of as a generalization of a vertex on a graph to an arbitrary CW complex. On a connected graph, the spanning co-trees are precisely the vertices, and $\psi_L([x])$ is the vertex itself.

2.3. Visualizing current generation. To describe the average current and the governing quantization results, we discuss[2] a simple example on S^1. We take the CW structure to have two 0-cells (vertices i and j), and two 1-cells (edges α and δ), as in Figure 2. Furthermore, assume we have taken both the adiabatic and low-temperature limits to simplify the discussion. We use a periodic driving protocol of good parameters, so that if at any time the vertex energies agree, the edge energies must be distinct, and vice-versa. In particular, take a periodic driving protocol γ, starting with $E_i < E_j$ and $W_\alpha < W_\delta$, split into 4 segments:

(i) vary E so that $E_i > E_j$, keeping W fixed,
(ii) vary W so that $W_\alpha > W_\delta$, keeping E fixed,
(iii) vary E so that $E_i < E_j$, keeping W fixed, and
(iv) vary W so that $W_\alpha < W_\delta$, keeping E fixed, and returning to the original parameters.

Initially, the particle will fall to vertex i, since it has the lowest energy. As E_i approaches E_j on segment (i), the particle will hop back and forth between i and j across the edge with lowest energy α. Once condition (i) is satisfied, the particle will sit at node j, and hence will have travelled a net distance of one across α and zero across δ. On segment (ii), since the vertex energies remain fixed, the particle will not move. Once E_i and E_j vary on segment (iii), the particle will hop back and forth between the vertices, this time crossing δ a net total of once, while not crossing α, and finishing at vertex i. No motion will occur on segment (iv), because the vertex energies are unchanged. Therefore, over one driving protocol, the particle will perform one full rotation around S^1. The homology class of this trajectory $Q(\gamma)$ clearly represents the generator of $H_1(S^1;\mathbb{Z}) \subset H_1(S^1;\mathbb{R})$ and the average current is 1. Since the average current lies in the integer lattice of

[2]This is a restatement of a discussion in [**CKS12**].

the real homology group, we say the current is *quantized*. This is the idea which stands behind the quantization results of current. Moreover, this phenomenon is generic for graphs, as shown in [**CKS13**].

The key idea which underlies this motion (and many phenomena in physics) is that, as the energies vary, the particle is always tending to or 'following' the vertex with lowest energy through the spanning tree of minimal energy, as seen in the motion on the graph of Figure 2. On segments (ii) and (iv), E has a unique minimum, and hence there is a preferred vertex or spanning co-tree in X. On segments (i) and (iii), W has a unique minimum, so we can construct a preferred spanning tree. In our example, this tree consists of a single edge, and contains both vertices. On a more complicated graph, the particle would traverse the unique path in T given by K^T_{j-i}, connecting vertices of minimal energy i and j.

The situation in higher dimensions is analogous. The cycle will jump from spanning co-tree to spanning co-tree by traversing spanning trees. The evolving cycle is always attempting to minimize $\sum_{b\in L} E_b$, thought of as the 'energy' of the cycle. As in the graph case, one can always form a decomposition of the periodic driving protocol γ to alternate between segments of type U, with a unique co-tree, and type V, with a unique tree. For example, consider a driving protocol γ going from type U to V to U, with associated spanning co-tree L, spanning tree T, and spanning co-tree L', respectively. On the U_L segment, the initial cycle tends to the cycle $\psi_L([\hat{x}])$ supported on the spanning co-tree L. The cycle remains in this configuration with overwhelming probability until the parameters change further. This occurs on the V_T segment, where the cycle will transition from $\psi_L([\hat{x}])$ to $\psi_{L'}([\hat{x}])$ within the spanning tree T. In fact, the transition occurs through the unique d-chain $K^T_{L-L'}$, and ends once the cycle becomes $\psi_{L'}([\hat{x}])$. These types of transitions keep happening until a full period of the driving protocol occurs.

The average current generated by this process can be written as a sum over segments alternating between type U and type V. On type U segments, the object will remain on the unique spanning co-tree and no current will be generated. On type V segments, current is generated by the motion along spanning trees. The coefficients which appear in the formula for the trajectory, or average current, $Q(\gamma)$ depend entirely on these subcomplexes to which the motion is restricted. The main result of [**CCK**] is that, in the long time limit, $Q(\gamma)$ will have rational incidence with each of the d-cells in X. That is, it will form a rational d-dimensional homology class, as opposed to a real homology class. This is in contrast to the main result on graphs [**CKS13**], in which the current is integer-valued, as in the previous example on S^1. The reason for this difference is due to a variety of factors, notably the order of torsion subgroups which are non-trivial in higher dimensions. This is further complicated by the more elaborate structures, like a vertex compared to a generic spanning co-tree.

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