Enchiridion: Mathematical User's Guides Volume 2 (September 2016)
`mathusersguides.com`

A user's guide: Completed power operations for Morava E-theory

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1. Key insights and central organizing principles

This user's guide is about the paper *Completed power operations for Morava E-theory*, written jointly with Tobias Barthel [**BF15**].

1.1. Background. Given an $\mathbb{E}_\infty$ (or more generally $\mathbb{H}_\infty$) ring spectrum E, *power operations* for E are the algebraic structure found in the homotopy of a commutative E-algebra. For $E = H\mathbb{F}_p$, the mod p Eilenberg-MacLane spectrum, power operations generalize the Dyer-Lashof operations on the mod p homology of infinite loop spaces. For $E = KU$, periodic complex K-theory, power operations on the homotopy of *p-complete* KU-algebras have been described in work of McClure and Bousfield. They are given by θ-algebras over the p-adic integers $\mathbb{Z}_p$, which are commutative $\mathbb{Z}_p$-algebras with some additional structure.

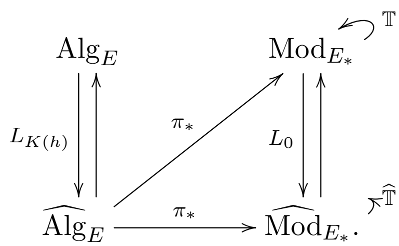
From now on, fix a prime number p and a height $h \geq 1$, and consider Morava E-theory $E = E_h$ at height h, and Morava K-theory $K(h)$. Power operations for E on the homotopy of $K(h)$-local commutative E-algebras are understood thanks to work of Ando, Hopkins, Strickland, and Kashiwabara. Building on this, Rezk constructed a monad $\mathbb{T}$ on the category Mod_{E_*} of E_*-modules which encodes the power operations [**Rez09**]. For example, at height $h = 1$, Morava E-theory is p-complete K-theory $E_1 = KU_p^\wedge$, and the monad $\mathbb{T}$ encodes θ-algebras.

Working $K(h)$-locally is crucial in those constructions. Now, an E-module spectrum X is $K(h)$-local if and only if its homotopy $\pi_* X$ is an E_*-module which is L-complete, a property studied notably in work of Greenlees, May, Hovey, and Strickland[1]. This provides the insight motivating the paper:

[1] More precisely, L-completion is the best approximation of $\mathfrak{m}$-adic completion by a right exact functor. The comparison map $L_0 M \to M_\mathfrak{m}^\wedge$ is an isomorphism for instance when the module M is finitely generated or flat. In those cases, being L-complete is the same as being $\mathfrak{m}$-adically complete.

> *If we only work with $K(h)$-local E-module spectra on the topological side, then we should only work with L-complete E_*-modules on the algebraic side.*

If A is a $K(h)$-local commutative E-algebra, then its homotopy $\pi_* A$ is an L-complete E_*-module which carries a $\mathbb{T}$-algebra structure. Since algebras over a monad have good categorical properties, it would be nice to encode the structure found in $\pi_* A$ via a monad. Namely, the monad $\mathbb{T}$ on Mod_{E_*} ought to be tightened to a monad $\widehat{\mathbb{T}}$ on the subcategory $\widehat{\mathrm{Mod}}_{E_*}$ of L-complete E_*-modules, so that $\pi_* A$ is naturally a $\widehat{\mathbb{T}}$-algebra. The following diagram explains the problem schematically:



Principle 1.1. *Monads are a convenient tool to encode additional structure, in topology as well as in algebra.*

In our situation, here is how monads encode the information on the topological side. Start with the free commutative E-algebra monad $\mathrm{Sym}\colon \mathrm{Mod}_E \to \mathrm{Mod}_E$ on the (model) category of E-modules, given by symmetric powers:

$$\mathrm{Sym}(M) = \bigvee_{n \geq 0} \mathrm{Sym}^n M = \bigvee_{n \geq 0} \left(M^{\wedge_E n} \right)_{\Sigma_n}.$$

Its derived functor $\mathbb{P}\colon h\mathrm{Mod}_E \to h\mathrm{Mod}_E$ becomes a monad on the homotopy category of E-modules, namely the free $\mathbb{H}_\infty$ E-algebra monad, given by *extended powers*:

$$\mathbb{P}(M) = \bigvee_{n \geq 0} \mathbb{P}_n M = \bigvee_{n \geq 0} \left(M^{\wedge_E n} \right)_{h\Sigma_n}.$$

The functor $\mathbb{T}_n\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ is constructed so that there is an approximation map

$$\mathbb{T}_n \left(\pi_* M \right) \to \pi_* L_{K(h)} \mathbb{P}_n M$$

for any E-module M, which is an isomorphism if M is finitely generated and free, or **finite free** for short.

The following table summarizes the ingredients of the problem, and how the main theorem of [**BF15**] improves the construction of the monad $\mathbb{T}$ to a monad $\widehat{\mathbb{T}}$ which takes L-completeness into account.

	Topology	Algebra
Ground ring	E	$E_* := \pi_* E$
Modules	Mod_E	Mod_{E_*}
Completion	$L_{K(h)} \colon \mathrm{Mod}_E \to \widehat{\mathrm{Mod}}_E$	$L_0 \colon \mathrm{Mod}_{E_*} \to \widehat{\mathrm{Mod}}_{E_*}$
Monad of interest	$\mathbb{P} = \bigvee_{n \geq 0} \mathbb{P}_n$	$\mathbb{T} := \bigoplus_{n \geq 0} \mathbb{T}_n$
Completed version	$\widehat{\mathbb{P}} := L_{K(h)} \mathbb{P}$	$\widehat{\mathbb{T}} = L_0 \mathbb{T}$
Compatibility	$L_{K(h)} \mathbb{P} \xrightarrow{\sim} L_{K(h)} \mathbb{P} L_{K(h)}$	$L_0 \mathbb{T} \xrightarrow{\text{iso?}} L_0 \mathbb{T} L_0$ **Main Theorem: Yes!**
Consequence	$\widehat{\mathbb{P}}$ is a monad on $h\widehat{\mathrm{Mod}}_E$.	$\widehat{\mathbb{T}}$ is a monad on $\widehat{\mathrm{Mod}}_{E_*}$.

There is a trade-off between algebraic invariants that encode more *structure*, which are more powerful but hard to compute, and crude invariants, which are less powerful but easier to compute. However, there is no downside to encoding more *properties*. As an analogy, if X is a space, then it is sometimes useful to view $\pi_2 X$ as a $\pi_1 X$-module, sometimes more convenient to view it as an abelian group. However, there would be no benefit in forgetting that $\pi_2 X$ is abelian and viewing it merely as a group.

PRINCIPLE 1.2. *Algebraic invariants of topological objects should encode as much structure as possible (or as convenient), and as many properties as possible.*

In the case of a $K(h)$-local commutative E-algebra A, one could look at all power operations on $\pi_* A$, encoded by a $\mathbb{T}$-algebra structure, or the underlying commutative E_*-algebra, or the underlying E_*-module, which only depends on the underlying E-module of A. In other words, there are forgetful functors:

$$\mathrm{Alg}_{\mathbb{T}} \longrightarrow \mathrm{Alg}_{E_*} \longrightarrow \mathrm{Mod}_{E_*},$$

along with L-complete analogues, where we require the underlying E_*-module to be L-complete. The underlying E_*-module of $\pi_* A$ is L-complete, and we want to remember that *property*.

1.2. Algebraic ideas. To prove the main theorem, our strategy is twofold: reduce the problem about E_*-modules to finite free modules, then prove the desired properties about E-modules, on the topological side. Here are the ingredients on the algebraic side.

KEY IDEA 1.3. *Reduce statements about L-complete E_*-modules to statements modulo the maximal ideal $\mathfrak{m} \subset E_*$.*

This idea is implemented by a Nakayama-type lemma [**BF15**, Lemma A.8], which provides a useful tool to detect isomorphisms between L-complete modules.

Using this, one then works with vector spaces over $E_*/\mathfrak{m}$, though the functors involved, such as $E_*/\mathfrak{m} \otimes \mathbb{T}_n$ are not additive.

KEY IDEA 1.4. *Reduce the problem to simpler E_*-modules and simpler E_*-module maps.*

This idea is implemented in [**BF15**, §4.1] using the following tricks.

- Any cokernel can be written (canonically) as a *reflexive* coequalizer. In particular, any module is a reflexive coequalizer of free modules.
- A free module is (canonically) a filtered colimit of finite free modules. More generally, a module is *flat* if and only if it is a filtered colimit of finite free modules.
- For a direct sum of modules $M \oplus N$, there is a "binomial formula"
$$\mathbb{T}_n(M \oplus N) = \bigoplus_{i+j=n} \mathbb{T}_i(M) \otimes \mathbb{T}_j(N).$$
This follows from the coproduct preservation formula $\mathbb{T}(M \oplus N) = \mathbb{T}(M) \otimes \mathbb{T}(N)$.
- The most relevant E_*-module maps are those given by multiplication by a scalar $\nu \in E_*$. In fact, it suffices to consider $\nu \in E_*$ taken from the (finite) set of generators of the maximal ideal $\mathfrak{m} \subset E_*$.

KEY IDEA 1.5. *Let F be a finite free E_*-module. Then an E_*-module map $\varphi \colon F \to F$ is nilpotent modulo the maximal ideal $\mathfrak{m} \subset E_*$ if and only if $\varphi^{-1}F$ is trivial modulo $\mathfrak{m}$, i.e., the following equality holds:*
$$E_*/\mathfrak{m} \otimes \operatorname{colim} \left(F \xrightarrow{\varphi} F \xrightarrow{\varphi} F \to \dots \right) = 0.$$

1.3. Topological ideas. The translation into topology relies on the fact that taking homotopy $\pi_* \colon \mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$ induces an equivalence $h\mathrm{Mod}_E^{\mathrm{ff}} \cong \mathrm{Mod}_{E_*}^{\mathrm{ff}}$ between the homotopy category of finite free E-modules and the category of finite free E_*-modules.

KEY IDEA 1.6. *Once the problem has been reduced to finite free modules, express algebraic problems about E_*-modules topologically in terms of E-modules.*

This strategy is carried out in [**BF15**, §4.2]. For this, we need topological analogues of some of the algebraic facts mentioned above.

- An E-module is flat if and only if it is a filtered colimit of finite free E-modules.
- On an E_*-module of the form $F = \pi_* M$, the E_*-module map $\nu \colon F \to F$ given by multiplication by a scalar $\nu \in E_*$ is the effect on homotopy of the E-module map $\nu \wedge_E \mathrm{id} \colon M \to M$.

- If $f\colon M \to M$ is an E-module map, then the algebraic construction $(\pi_* f)^{-1}(\pi_* M)$ is the homotopy of the mapping telescope
$$f^{-1}M = \operatorname{colim}\left(M \xrightarrow{f} M \xrightarrow{f} M \to \dots \right)$$
since π_* preserves filtered colimits.

1.4. Side issues. In [**Rez09**], treating the $\mathbb{Z}$-grading required significant work and care. In [**BF15**], the grading does not play an important role. Most of the argument can be made over the ring E_0. Minor adaptations then yield the result over the graded ring E_*, using the fact that E_* is 2-periodic.

References

[BF15] Tobias Barthel and Martin Frankland, *Completed power operations for Morava E-theory*, Algebr. Geom. Topol. **15** (2015), no. 4, 2065–2131.

[Rez09] Charles Rezk, *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327 –379.

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