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A user's guide: Completed power operations for Morava E-theory

Martin Frankland

2. Metaphors and imagery

2.1. Rings and modules. The paper relies on the fact that Morava E-theory is a highly structured ring spectrum, namely, an $\mathbb{E}_\infty$ ring spectrum. An $\mathbb{E}_1$ (or equivalently, $\mathbb{A}_\infty$) ring structure guarantees that categories of module spectra behave well homotopically, while an $\mathbb{E}_\infty$ ring structure guarantees that the smash product $\wedge_E$ of E-modules behaves well. Since the advent of models for spectra with a strictly associative smash product, notably S-modules [**EKMM97**] and symmetric spectra [**HSS00**], we can do stable homotopy theory in a way that mimics algebra. As far as I understand, this was part of the impetus behind "brave new rings"—an expression coined by Waldhausen [**Gre07**]—and higher algebra.

The metaphor is this: *Ring spectra and module spectra are like ordinary rings and modules... but trickier.* At least the formal manipulations are similar. In our paper, we deal with notions for modules such as free, finitely generated, finitely presented (or perfect), and flat, as well as module maps given by multiplication by a scalar, a scalar acting invertibly on a module, etc. For these notions, the analogy between topology and algebra works remarkably well. In fact, our extension of Lazard's flatness criterion to module spectra [**BF15**, §2.2] was motivated by this analogy, and in turn strengthens it.

As useful as it may be, the analogy has its limitations. We can think of extended powers $\mathbb{P}_n M = (M^{\wedge n})_{h\Sigma_n}$ as the analogue of symmetric powers in algebra $\mathrm{Sym}^n M = (M^{\otimes n})_{\Sigma_n}$. However, the homotopical construction is richer. For the monad $\mathbb{T}\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ described in Section 1.1, every $\mathbb{T}$-algebra has an underlying commutative E_*-algebra, and the decomposition $\mathbb{T} = \oplus_{n\geq 0}\mathbb{T}_n$ corresponds to the decomposition $\mathrm{Sym} = \oplus_{n\geq 0} \mathrm{Sym}^n$. There is a natural comparison map $\mathrm{Sym}^n M \to \mathbb{T}_n M$ which is rarely an isomorphism, as $\mathbb{T}_n$ encodes additional, more exciting structure. For instance, at height $h = 1$, the monad $\mathbb{T}$ is related to θ-rings; see [**BF15**, Theorem 6.14] for a precise statement. A θ-ring is a commutative ring R equipped with a non-linear operation $\theta\colon R \to R$ satisfying certain equations. The operation θ recovers the Adams operation $\psi\colon R \to R$ via the formula $\psi(x) = x^p + p\theta(x)$.

Since the paper is about compatibility of the functors $\mathbb{T}_n$ with L-completion, let us describe metaphors related to L-completion, and then come back to the functors $\mathbb{T}_n$.

2.2. The p-adic topology and power series. As mentioned previously, L-completion is closely related to $\mathfrak{m}$-adic completion, and they often agree. At height $h = 1$, the maximal ideal is $\mathfrak{m} = (p) \subset E_* = \mathbb{Z}_p[u^{\pm}]$, where $\mathbb{Z}_p$ denotes the p-adic integers. Here, let us ignore the 2-periodicity of E_* and focus on the degree zero part $E_0 = \mathbb{Z}_p$. Thus, L-completion is a variant of p-adic completion $M^{\wedge}_p = \lim_k M/p^k M$.

In the p-adic topology, p is viewed as small, and higher powers p^k are even smaller. Of course, this is unrelated to the "size" of p^k as an integer number. As an analogy, think of the behavior of polynomials or power series in x as x tends to 0. In that situation, x is viewed as a small quantity, and x^{100} is much smaller. The behavior of the power series is dictated by the term with lowest exponent, for example:

$$f(x) = \overbrace{x^5}^{\text{dominant term}} + \overbrace{x^6 + 10x^{23} + 3x^{40} + \dots}^{\text{negligible}}$$

Replacing x by p everywhere, we can say the same about the elements of $\mathbb{Z}_p$, and more generally within an L-complete $\mathbb{Z}_p$-module.

It turns out that this analogy can be made rigorous. In [**Rez13**], Rezk shows that L-completion is given by *analytic p-completion*, in the following sense:

$$L_0 M \cong \operatorname{coker}\left(M[\![x]\!] \xrightarrow{(x-p)} M[\![x]\!] \right).$$

One can then show that in an L-complete abelian group $M = L_0 M$, every element $f \in M$ admits a "Taylor expansion around p", which is a certain power series $\sum_n c_k (x-p)^k \in M[\![x]\!]$ representing f. An analogous description holds at higher height, where the maximal ideal $\mathfrak{m}$ has h generators instead of one. L-completion is then analytic completion with respect to h variables.

2.3. Controlling error terms. In a given L-complete module M, p is "small" and p^k is "close" to 0 if k is large. Given a map of modules $f\colon M \to N$, one can ask how continuous f is with respect to the p-adic topology. Now we are interested in the functor $\mathbb{T}_n\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ and we push the analogy one level up. Think of the quotient map $q\colon M \twoheadrightarrow M/p^k M$ as being "close" to an isomorphism when k is large. In our paper, the main step towards the main theorem is to show that the induced map

$$E_*/\mathfrak{m} \otimes \mathbb{T}_n(E_*) \xrightarrow{E_*/\mathfrak{m} \otimes \mathbb{T}_n(q)} E_*/\mathfrak{m} \otimes \mathbb{T}_n(E_*/\mathfrak{m}^k) \tag{1}$$

is an isomorphism for k large enough [**BF15**, Proposition 4.1].

Here is an analogy with first-year calculus. Like many people, I think of the ϵ–δ definition of a limit $\lim_{x \to a} f(x) = L$ as a challenge and response. The ϵ is the challenge, while $\delta = \delta(\epsilon)$ is my response to that challenge. I don't expect $f(x) = L$ to hold anywhere, but as long as I can bound the error term $|f(x) - L|$ by ϵ, then I'm done.

In Equation (1), the fact that we are tensoring outside by the residue field $E_*/\mathfrak{m}$ is akin to a given challenge ϵ. We could have tensored outside by $E_*/\mathfrak{m}^{100}$ for a more difficult challenge, akin to taking a smaller ϵ, but in this situation, it turns out that tensoring outside by $E_*/\mathfrak{m}$ is enough, by a Nakayama-type argument. Our response, akin to finding a small enough δ, is to find an exponent k large enough such that the "small" map $E_* \to E_*/\mathfrak{m}^k$ is sent to the map $\mathbb{T}_n(E_*) \to \mathbb{T}_n(E_*/\mathfrak{m}^k)$ which is not quite an isomorphism, but "small enough" that it becomes an isomorphism after tensoring with $E_*/\mathfrak{m}$.

Note that if $\mathbb{T}_n$ commuted with $E_*/\mathfrak{m} \otimes -$, then the exponent $k = 1$ would work and we'd be done. The functor Sym^n is an example of functor that commutes with $E_*/\mathfrak{m} \otimes -$. One can show that $\mathbb{T}_n$ agrees with Sym^n (on finitely generated modules) for $n < p$. Hence, the first interesting example, where an exponent $k > 1$ might be required, is $\mathbb{T}_p$.

Example 2.1. *At height $h = 1$, the quotient map $q\colon \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^k$ induces the map*

$$\begin{array}{ccc} \mathbb{T}_p(\mathbb{Z}_p) & \xrightarrow{\mathbb{T}_p(q)} & \mathbb{T}_p(\mathbb{Z}/p^k) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbb{Z}_p \oplus \mathbb{Z}_p & \twoheadrightarrow & \mathbb{Z}/p^{k+1} \oplus \mathbb{Z}/p^{k-1} \end{array}$$

where the bottom map is the direct sum of the two quotient maps. Therefore, the map $\mathbb{Z}/p \otimes \mathbb{T}_p(q)$ is an isomorphism for all $k \geq 2$. The following table illustrates the thought process in a slightly cartoonish way.

k	$\mathbb{Z}/p \otimes \mathbb{T}_p(q)$	*My thought*
$k = 1$	$\mathbb{Z}/p \oplus \mathbb{Z}/p \twoheadrightarrow \mathbb{Z}/p$	*Danger! Not yet injective.*
$k = 2$	$\mathbb{Z}/p \oplus \mathbb{Z}/p \xrightarrow{\simeq} \mathbb{Z}/p \oplus \mathbb{Z}/p$	✓ *Safe. The map is now injective.*
$k =$ *bajillion*	$\mathbb{Z}/p \oplus \mathbb{Z}/p \xrightarrow{\simeq} \mathbb{Z}/p \oplus \mathbb{Z}/p$	✓ *Even safer.*

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Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany

E-mail address: martin.frankland@uni-osnabrueck.de