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A user's guide: Completed power operations for Morava E-theory

Martin Frankland

3. Story of the development

While at Urbana-Champaign, I received an FQRNT Postdoctoral Fellowship starting in the fall 2011, to work under the supervision of Charles Rezk. In [**Rez09**, §1.6], Charles had raised a question for future research, which he suggested to me as a problem to work on, as it was a good fit for my interests. This problem became the topic of [**BF15**]. At first, I had to learn some background material on chromatic homotopy theory, Morava E-theory, power operations, θ-rings, λ-rings, L-complete modules, and so on. Charles' guidance was immensely helpful throughout the process.

At the beginning, I focused on the case of height $h = 1$, in which case the monad $\mathbb{T}$ admits an explicit description, and thus the problem can be tackled using explicit formulas. Already at height 1, the Nakayama-type reduction [**BF15**, Lemma A.8] was one of the first steps. Here's the reason: The goal was to show that the map

$$\mathbb{T}_n(\eta)\colon \mathbb{T}_n M \to \mathbb{T}_n L_0 M$$

induces an isomorphism upon applying L-completion L_0. The reduction says that under mild assumptions, it suffices to check this modulo the maximal ideal $(p) \subset \mathbb{Z}_p$. It was also clear that controlling the torsion in $\mathbb{T}_n(\mathbb{Z}/p^k)$ was a key issue, and that this module is computed by turning the cokernel diagram

$$\mathbb{Z}_p \xrightarrow{p^k} \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^k$$

into a reflexive coequalizer.

At the "Strings and Automorphic Forms in Topology" conference in Bochum in August 2012, I presented the result for $h = 1$ in a contributed talk. My talk caught the attention of Tobias Barthel, then a graduate student at Harvard, who had been thinking about related topics. Thus, we started discussing on that occasion. At the Quillen Memorial Conference at MIT in October 2012, I gave a similar talk in a discussion session. There, I had a chance to discuss the project with Mark Hovey, Mark Behrens, and Haynes Miller, and talk some more to Tobias. Later in October, at the Midwest Topology Seminar at Michigan State University, Tobias and I discussed how my work at height $h = 1$, along with the use of flat modules, could reduce the problem at arbitrary height h to a key technical property of the functors $\mathbb{T}_n$; this reduction step later became [**BF15**, §4.1]. That is when we "officially" joined forces to solve the general case $h \geq 1$.

In parallel, we had been wondering why Charles' proof that $\mathbb{T}$ is a monad does not also show formally that $\widehat{\mathbb{T}}$ is a monad. Looking at his proof more closely, we pinpointed where the argument breaks down for $\widehat{\mathbb{T}}$, because of peculiar features of the category $\widehat{\mathrm{Mod}}_{E_*}$ of L-complete E_*-modules. Specifically, E_* is small as an object of Mod_{E_*} but *not* as an object of $\widehat{\mathrm{Mod}}_{E_*}$; c.f. [**BF15**, Remark 3.21]. This spurred us to study L-complete modules more thoroughly. Many results had already been worked out in [**HS99**, Appendix A]. Another important ingredient was found in unpublished notes of Hovey: L-completion preserves flatness [**BF15**, Proposition A.15]. We collected useful facts about L-complete modules into [**BF15**, Appendix A].

Proving the key property of $\mathbb{T}_n$ involved some reverse engineering. Looking at how $\mathbb{T}_n(E_*/\mathfrak{m}^k)$ is computed as a reflexive coequalizer, we saw that the important ingredient was nilpotency of the map $\mathbb{T}_n(\nu)\colon \mathbb{T}_n M \to \mathbb{T}_n M$ modulo the maximal ideal $\mathfrak{m} \subset E_*$, for a scalar $\nu \in E_*$. This in turn could be proved by going back to the topological side and using a mapping telescope [**BF15**, Corollary 4.5]. Charles' input also helped us in that section, notably in the proof of [**BF15**, Lemma 4.4]. For instance, he reminded us that a scalar $\nu \in E_*$ acts invertibly on an E-module M if and only if the natural map

$$M = E \wedge_E M \to (\nu^{-1}E) \wedge_E M = \nu^{-1}M$$

is an equivalence.

Working with flat E_*-modules made us think about flat E-module spectra. Helpful email discussions with Charles in August 2013 led to Lazard's criterion for flat module spectra [**BF15**, §2.2], which we use in the proof of [**BF15**, Corollary 4.5]. Said discussions were motivated by the following. We were comparing some work of Charles on flat modules over the (periodic) ring spectrum E with work of Jacob Lurie on flat modules over *connective* ring spectra. We wanted to clarify the relationship between the two approaches, and to find a common generalization; c.f. [**BF15**, Remark 2.11].

In [**BF15**, §6], we describe the monad $\mathbb{T}$ at height $h = 1$. This result is known to experts and is due to Jim McClure [**BMMS86**, §IX]. However, it is not obvious how the calculations of McClure translate into the description of $\mathbb{T}$, and how this relates to work of Bousfield on θ-rings [**Bou96**], which is why we spelled out the details in our paper. Much of that section results from helpful email discussions with Jim McClure, spread out from October 2013 to February 2014. The first round of exchanges made it into the first arXiv version, posted in November 2013. Subsequent exchanges yielded the main changes in version 2, which was posted in February 2014. Notably, it was Jim McClure who suggested viewing his operation Q as an operation on the homotopy of (p-complete) $\mathbb{H}_\infty$ KU-algebras rather than on the (completed) KU-homology of $\mathbb{H}_\infty$ ring spectra [**BF15**, Proposition 6.8].

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Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany

E-mail address: `martin.frankland@uni-osnabrueck.de`