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A user's guide: Completed power operations for Morava E-theory

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4. Colloquial summary

I will start with a gentle introduction to topology. Then I will leave a black box around the specifics of our paper [**BF15**] and focus only on one aspect: the word "completed" in the title.

4.1. What is topology? Topology is the study of spaces, such as curves, surfaces, and higher-dimensional analogues. Unlike geometry, which cares about angles, lengths, and volume, topology looks at the qualitative features of a space, its general shape: how many pieces there are, whether there are holes, the number of holes, and so on. That's why topology is sometimes called rubber-sheet geometry: spaces can be stretched, compressed, bent, and it's all the same to us. According to a classic joke, a topologist cannot distinguish a donut from a coffee mug (Figure 1).

Likewise, a balloon, the surface of a football, or the surface of the Earth are all the same space, namely a 2-dimensional sphere. However, the donut and the sphere are different spaces (Figure 2).

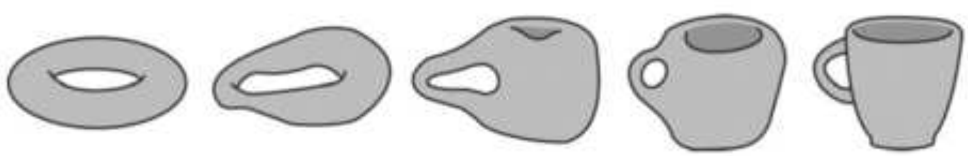

FIGURE 1. A donut and a coffee mug are equivalent spaces.
Image credit: Wikimedia Commons, via www.functionspace.com.

Algebraic topology is the branch of topology that describes the shape of spaces using algebraic invariants, quantities that we can compute. For example, using algebra, we can make precise the idea that the donut has a hole in it that the sphere doesn't have. For another example, a loop in a circle is described by

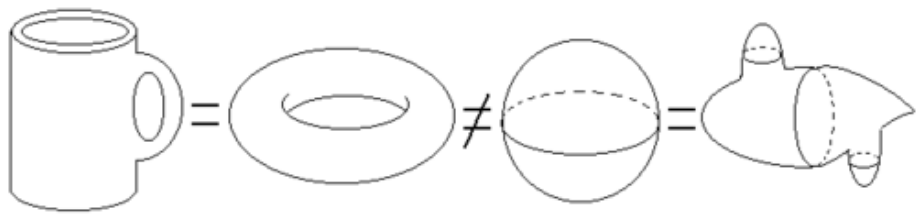

FIGURE 2. A donut and a sphere are *not* equivalent spaces.
Image credit: www.functionspace.com.

the number of times it's winding around the circle (Figure 3). By counting the winding number, we describe topological information (the loops in a circle) via an algebraic structure (the integer numbers $\mathbb{Z}$).

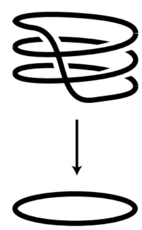

FIGURE 3. A loop winding 3 times around the circle.
Image credit: Allen Hatcher, *Algebraic Topology*, Cambridge University Press.

As exciting as spaces are, you may wonder what they're good for. Topology has become an important branch of mathematics, with connections to geometry, analysis, algebra, and mathematical physics. Since the 1990s, algebraic topology has also been applied to problems such as data analysis, sensor networks, and robot motion planning.

4.2. What is completeness? The paper [**BF15**] takes place in a branch of algebraic topology called *chromatic homotopy theory*. We study certain kinds of "spaces" and their algebraic invariants, which have the interesting feature of being *complete*. Our main theorem says that certain computations can be made using only algebraic structures that are complete. In this section, I will sketch what "complete" means, and why it might be an interesting feature.

Consider the number $\sqrt{2}$, which you may have seen in a geometry class (Figure 4).

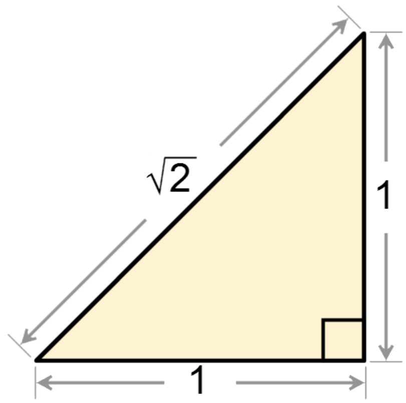

FIGURE 4. A right triangle.
Image credit: Wikimedia Commons.

Its decimal expansion is

$$\begin{aligned}\sqrt{2} &= 1.41421\ldots \\ &= 1 + 0.4 + 0.01 + 0.004 + 0.0002 + 0.00001 + \ldots \\ &= 1 \cdot 10^0 + 4 \cdot 10^{-1} + 1 \cdot 10^{-2} + 4 \cdot 10^{-3} + 2 \cdot 10^{-4} + 1 \cdot 10^{-5} + \ldots\end{aligned}$$

Recall that $\sqrt{2}$ is irrational, i.e., cannot be expressed as a fraction[1]. Another way to say this is that the decimal expansion of $\sqrt{2}$ does not start repeating after a while, which is what happens with a fraction, e.g.:

$$\frac{5}{6} = 0.8\overline{3} = 0.833333333\ldots$$

The rational numbers $\mathbb{Q}$ are *incomplete*: there is a gap where $\sqrt{2}$ would be expected. In contrast, the real numbers $\mathbb{R}$ are *complete*: they contain all numbers obtained from decimal expansions.

Another example of infinite sum appears in calculus, when expressing functions as their *Taylor series*[2]. For instance, the exponential function is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

This infinite sum is useful both in theory and in practice. To this day, some calculators use this formula to compute the exponential function, by adding the first few terms, depending how much precision we need. Take for instance $x = 0.1$,

[1]This fact was discovered in Ancien Greece in the 5th century BC, sometimes attributed to Hippasus of Metapontum. According to legend, he was murdered for his discovery of irrational numbers, deemed shocking at the time by his fellow Pythagoreans.

[2]Named after the English mathematician Brook Taylor, who introduced them formally in 1715. Some cases had already been used by Isaac Newton and James Gregory in the 1660s, and by Indian mathematician Madhava in the 14th century.

and compare these values:

$$e^{0.1} \approx 1.105171$$
$$1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} \approx 1.105167.$$

The first few terms already yield a good approximation.

In our paper, the word "complete" means that certain infinite sums are available, which is convenient. Let's say we compute something involving the algebraic invariants of our spaces. If we insist on a certain precision in the answer, we know we can achieve it by taking enough terms of the infinite sum.

Here's a non-mathematical analogy as to why completeness is useful. We topologists study spaces, as an ornithologist studies birds, or a botanist studies plants. The algebraic invariants we associate to a space provide some information about the space, as pictures of birds or plants provide some information about them. Having algebraic invariants that are complete would be like having high-resolution pictures, where one can zoom in and still see a clear picture.

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