Enchiridion: Mathematical User's Guides Volume 2 (September 2016)
mathusersguides.com

A user's guide: Landweber flat real pairs and $ER(n)$ cohomology

Vitaly Lorman

This user's guide is about the paper *Landweber flat real pairs and $ER(n)$-cohomology* [**KLW16**], joint with Nitu Kitchloo and W. Stephen Wilson.

1. Key insights and central organizing principles

1.1. Introduction. Algebraic topologists view the world (topological spaces) through the lens of algebraic invariants (cohomology theories). Two criteria along which cohomology theories are often judged are

(i) how much information they see and (ii) how computable they are.

These two criteria often pull in opposite directions. Given a generic cohomology theory, the first spaces one might try to compute its value on are some building blocks such as spheres S^k, projective spaces $\mathbb{RP}^k$ or $\mathbb{CP}^k$, Eilenberg MacLane spaces $K(G, q)$, or classifying spaces of various compact Lie groups BG. The goal of [**KLW16**] is to compute the value of the cohomology theory Real Johnson-Wilson theory, $ER(n)$, on some of these spaces (about half of them). Before we get into the details, let's step back and build a case for why $ER(n)$ is a theory worth investigating, i.e. why it ranks highly in both criteria.

1.2. Detecting torsion. Computing the stable homotopy groups of spheres is a major motivating problem in algebraic topology. One way to get a handle on elements in π_*S^0 is to detect them using various cohomology theories. That is, given a multiplicative cohomology theory E, we may hope that some elements of π_*S^0 show up in the coefficients E_* via the image of the unit map $\pi_*S^0 \longrightarrow E_*$ We could then study how these spherical classes act on $E^*(X)$ for various spaces X.

Positive degree elements of the stable homotopy groups of spheres are all torsion. As such, for the above approach to yield any interesting information, it is necessary that some of these classes do not map to zero in E_*. This, for one, requires that E_* contain torsion and furthermore that the unit map not factor though any cohomology theories whose coefficients are torsion-free. This latter criterion rules out any *complex-oriented* cohomology theory, as for all such theories, the unit map factors through MU_*. Unfortunately, by far the most computationally accessible cohomology theories to date have been the complex-oriented ones[1]. So, how do we compute with non complex-oriented theories?

EXAMPLE 1. (**KO and KU**) Complex K-theory KU is complex-oriented, torsion-free, and very amenable to computations. It also carries an action of the group of order 2, C_2, which can be seen geometrically via the action of complex conjugation on vector bundles. Taking fixed points produces real K-theory, KO, which is no longer complex-oriented or torsion-free. In fact, KO detects the spherical class η in its Hurewicz image. What makes KO computable is that η generates *all* of the torsion and so KO may be computed from KU by building in η^r-torsion one r at a time using the η-Bockstein spectral sequence $E_1^{*,*} = KU^*(Z) \Rightarrow KO^*(Z)$.

This example supports the claim that KO is a cohomology theory which satisfies both conditions laid out at the beginning. Furthermore, it suggests we can find more such theories by looking for complex-oriented cohomology theories which carry group actions and taking their fixed points. Chromatic homotopy theory, a subfield of algebraic topology, provides many examples of complex-oriented cohomology theories, from which we pick a certain family. From here on, we work 2-locally. The examples in [**KLW16**] are the Johnson-Wilson theories $E(n)$ [**JW73**] for $n \geq 1$. Their coefficients are torsion-free, given by

$$E(n)_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}], \quad |v_i| = 2(2^i - 1)$$

They all carry an action of C_2 which ultimately stems from complex conjugation, as with KU. In fact, $E(1)$ is $KU_{(2)}$.

Equivariant interlude To distinguish a cohomology theory with C_2-action from an ordinary nonequivariant one, we will denote the former with boldface letters, such as $\mathbb{E}(n)$. As we will see later, there will be great value in letting our group act on the spheres. As such, given a (real) representation V, let S^V denote the one-point compactification as an equivariant space. In the case of C_2, all representations have the form $s + t\alpha$ where α denotes the trivial representation. We may now index our homotopy and cohomology groups not just over integers but over representations.[a] Define $\pi_{s+t\alpha}(\mathbb{E}(n)) := [S^{s+t\alpha}, \mathbb{E}(n)]^{C_2}$ and $\mathbb{E}^{s+t\alpha}(X) = [X, \Sigma^{s+t\alpha}\mathbb{E}]^{C_2}$.

[a] Example: the regular representation, $1+\alpha$ is exactly the complex plane with C_2 acting by complex conjugation, and $S^{1+\alpha}$ is the Riemann sphere.

[1] The complex orientation implies that E has a theory of Chern classes and in particular makes possible the immediate computation of spaces like $\mathbb{C}P^\infty$, $\mathbb{CP}^k$, $BU(q)$, and $B\mathbb{Z}/(p^k)$. There are many other computations that take more work. See e.g. [**RW80**, **RWY98**] for the complex-oriented cohomology of Eilenberg-MacLane spaces and much more.

As with KU, we may take the C_2-fixed points of $\mathbb{E}(n)$ to produce a family of interesting new cohomology theories.

Definition 1. Define *Real Johnson-Wilson theory* to be the C_2-fixed points of Johnson-Wilson theory, $ER(n) = \mathbb{E}(n)^{C_2}$.

Just as with KU and KO, it turns out that $ER(n)$ is close enough to $E(n)$ to be computationally accessible but has just enough extra information (via torsion) to make it interesting.

Theorem 1. *(Kitchloo-Wilson [**KW07a**]) There is a class $x \in \pi_\lambda ER(n)$ with $\lambda = 2^{2n+1} - 2^{n+2} + 1$ and a fibration $ER(n) \xrightarrow{x} ER(n) \longrightarrow E(n)$ which yields a Bockstein spectral sequence $E_1^{*,*} = E(n)^*(Z) \Rightarrow ER(n)^*(Z)$.*

Key Idea 1.1. *All of the torsion in the coefficients of $ER(n)$ is generated by a single class x. Thus, we may compute $ER(n)$-cohomology from $E(n)$-cohomology by building in x^r torsion for a single r at a time, as carried out by the Bockstein spectral sequence (BSS).*

1.3. Producing permanent cycles. We now have a potentially interesting cohomology theory, $ER(n)$, and a method to compute it from a more accessible cohomology theory, $E(n)$. The next step is to compute its value on some interesting spaces. We will describe the approach taken in [**KLW16**], first via an example from [**KW08**].[2].

Example 2. ($ER(n)^*(\mathbb{R}P^\infty)$**, part I**) We have a spectral sequence which begins with $E_1^{*,*} = E(n)^*(\mathbb{R}P^\infty)$ and converges to $ER(n)^*(\mathbb{R}P^\infty)$. The $E(n)$-cohomology of $\mathbb{R}P^\infty$ is known[a] and so the next step is to compute Bockstein differentials. While there is an explicit formula for d_1, computing higher differentials is difficult. The computation becomes easier if we can identify some classes in $E(n)^*(\mathbb{R}P^\infty)$ which do not support differentials (i.e. permanent cycles). These are classes in $E_1^{*,*} = E(n)^*(\mathbb{R}P^\infty)$ which are in the image of the map $ER(n)^*(\mathbb{R}P^\infty) \longrightarrow E(n)^*(\mathbb{R}P^\infty)$.
A crucial observation is that both $\mathbb{R}P^\infty$ and $ER(n)$ are fixed points—that is, $\mathbb{R}P^\infty = \mathbb{C}P^{\infty C_2}$ and $ER(n) = \mathbb{E}(n)^{C_2}$. Hence, given an equivariant map $\mathbb{C}P^\infty \longrightarrow \mathbb{E}(n)$, we may take fixed points to get a map $\mathbb{R}P^\infty \longrightarrow ER(n)$. In other words, there is a homomorphism

$$[\mathbb{C}P^\infty, \mathbb{E}(n)]^{C_2} \longrightarrow [\mathbb{R}P^\infty, ER(n)] = ER(n)^*(\mathbb{R}P^\infty)$$

[a] $E(n)^*(\mathbb{R}P^\infty) = E(n)^*(B\mathbb{Z}/2) = E(n)^*[u]/([2]_F(u))$ where $[2]_F(u)$ is a certain power series computable using the formal group law for $E(n)$

[2] The $ER(n)$ cohomology of projective spaces is not just a curiosity. In [**KW08**], Kitchloo and Wilson used their computation of $ER(2)$-cohomology to demonstrate some nonimmersion results for real projective spaces—for example, that $\mathbb{RP}^{48}$ does not immerse in $\mathbb{R}^{84}$. The torsion in $ER(2)$ is key to their results which are undetectable by any complex-oriented theory.

This example can be summarized in the following key idea:

Key Idea 1.2. *To get ahold of classes in $ER(n)^*(Z)$, map Z into the fixed points of another space Y. Then the image of the homomorphism*

$$[Y, \mathbb{E}(n)]^{C_2} \longrightarrow [Y^{C_2}, ER(n)] \longrightarrow [Z, ER(n)] = ER(n)^*(Z)$$

provides classes in $ER(n)^*(Z)$ which are detected by permanent cycles in the Bockstein spectral sequence.

In the above example, we have $(Z, Y) = (\mathbb{R}P^\infty, \mathbb{C}P^\infty)$ and the map from $\mathbb{R}P^\infty$ to the fixed points of $\mathbb{C}P^\infty$ happens to be an equivalence, though in general this is not a requirement. We have now reduced the problem to computing the source of the composite homomorphism above, the equivariant $E(n)$-cohomology of $\mathbb{C}P^\infty$.[3]

1.4. Computing equivariant cohomology.

The computation of

$$[\mathbb{C}P^\infty, \mathbb{E}(n)]^{C_2} = \mathbb{E}(n)^{*+*\alpha}(\mathbb{C}P^\infty)$$

has two conceptual ingredients: *the projective property* and *the hat construction.*

1.4.1. ***The projective property.*** A key input that makes our C_2-equivariant computations possible is that our spaces Y above (such as $\mathbb{C}P^\infty$) have an important property with respect to $\mathbb{E}(n)$, first defined by Kitchloo and Wilson in [**KW07b**]. We will work through what this means in the case of $\mathbb{C}P^\infty$. We begin by noting that the forgetful map $\rho : \mathbb{E}(n)_{*(1+\alpha)} \longrightarrow E(n)_{2*}$ is an isomorphism [**HK01**]. From this, we may filter the C_2-CW complex $\mathbb{C}P^\infty$ skeletally and notice that the associated graded is given by $\mathbb{CP}^k/\mathbb{CP}^{k-1} = S^{k(1+\alpha)}$. It follows that the forgetful map ρ is an isomorphism on these pieces. Furthermore, the $\mathbb{E}(n)$-cohomology of these quotients forms the E_2-page of an equivariant Atiyah-Hirzebruch spectral sequence converging to $\mathbb{E}(n)^{*+*\alpha}(\mathbb{C}P^\infty)$. This spectral sequence turns out to collapse (for reasons similar to those in the nonequivariant case), and the isomorphism above may be boosted up to an isomorphism

$$\mathbb{E}(n)^{*(1+\alpha)}(\mathbb{C}P^\infty) \xrightarrow[\rho]{\cong} E(n)^{2*}(\mathbb{C}P^\infty)$$

So we can at least compute the 'diagonal', degrees $k(1+\alpha)$ part of $\mathbb{E}(n)^\star(\mathbb{C}P^\infty)$. As we will see from the hat construction, computing the diagonal degrees is enough to give us everything we need. Definition 1.4 of [**KLW16**] defines what it means for spaces to be 'sufficiently projective' such that ρ is an isomorphism for them as well (proved in Theorem 1.5). Examples of spaces with this projective property include $\mathbb{C}P^\infty$, BU, BSU, $BU\langle 6\rangle$, and all Wilson spaces[4].

[3]Recall that this is a bigraded object. We are especially interested in classes in integral degrees $* + 0\alpha$.

[4]Wilson spaces are H-spaces whose p-local homotopy and homology are both free $\mathbb{Z}_{(2)}$-modules. In [**KLW16**], they are the projective spaces Y whose fixed points we map into to compute the $ER(n)$-cohomology of some Eilenberg MacLane spaces. For more of their properties, see Wilson's thesis [**Wil73, Wil75**] as well as [**RW77**] (and [**BW07**] for a generalization)

1.4.2. ***The hat construction***. But we have not computed all of the bigraded ring $\mathbb{E}(n)^{*+*\alpha}(\mathbb{C}P^\infty)$, and what we're really interested in are classes in integer degrees since those are the ones which will map to $ER(n)^*(\mathbb{R}P^\infty)$. It turns out that for computing $\mathbb{E}(n)$-cohomology, the diagonal degrees are enough. As observed by Kitchloo and Wilson, the coefficients $\mathbb{E}(n)_\star$ contain a class y in degree $\lambda+\alpha$, where λ is the same integer as above, and *this class is invertible!* It follows that, given any class z in degree $s+t\alpha$, we may define the product $\widehat{z} := zy^{-t}$, which lives in an integral degree. This defines an injective ring (but not graded ring) endomorphism of $\mathbb{E}(n)^\star(-)$ given by 'hatting' each class, and allows us to move all diagonal classes to integral degrees.

KEY IDEA 1.3. *Given Y and Z as above, classes in $E(n)^*(Y)$ can transform into classes in $ER(n)^*(Z)$ as follows. Combining the projective property and the hat construction, we have*

$$E(n)^{2*}(Y) \xrightarrow{\rho^{-1}} \mathbb{E}(n)^{*(1+\alpha)}(Y) \xrightarrow{\widehat{\ }} \mathbb{E}(n)^{*(1-\lambda)}(Y)$$

We may then postcompose with the map induced by $Z \longrightarrow Y^{C_2} \longrightarrow Y$ to get a map

$$\psi : E(n)^{2*}(Y) \longrightarrow ER(n)^{*(1-\lambda)}(Z)$$

The source is computable and so this map produces classes in the target that are easily detected as permanent cycles on the E_1-page of the BSS.

1.5. Putting the pieces together. We return to finish off our example.

EXAMPLE 3. ($ER(n)^*(\mathbb{R}P^\infty)$**, part II**) We have

$$E(n)^*(\mathbb{C}P^\infty) = E(n)^*[[u]].$$

We denote the image of the map ψ above by $\widehat{E(n)}^*(\mathbb{R}P^\infty)$; it is a subalgebra of $ER(n)^*(\mathbb{R}P^\infty)$. The E_1 page of the BSS is

$$E(n)^*(\mathbb{R}P^\infty) = E(n)^*[[u]]/([2]_F(u))$$

and $\widehat{E(n)}^*(\mathbb{R}P^\infty)$ maps to give us permanent cycles in $E_1^{*,*}$. In fact, these permanent cycles generate enough of the E_1-page such that knowledge of the differentials on the coefficients finishes the job. Concretely, let

$$\widehat{E(n)}^* = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, \widehat{v}_n^{\pm 1}]$$

where

$$\widehat{v}_i = v_i v_n^{(2^i-1)(2^n-1)}$$

is the result of applying ψ to $v_i \in E(n)^*$ and then mapping into $E(n)^*(\mathbb{R}P^\infty)$. Then it turns out that

$$E_1^{*,*} = E(n)^* \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty) = E_1^{*,*}(pt) \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty)$$

Note that, by construction, the right hand component of the tensor product consists entirely of permanent cycles. It thus turns out that the entire spectral sequence plays out on left hand component—that is, on the coefficients—where we know all of the differentials[a]. In particular, we thus have

$$E_\infty^{*,*} = E_\infty^{*,*}(pt) \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty)$$

and so

$$ER(n)^*(\mathbb{R}P^\infty) = ER(n)^* \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty)$$

Recall that $\widehat{E(n)}^*(\mathbb{R}P^\infty)$ is isomorphic to $E(n)^*(\mathbb{R}P^\infty)$. Thus, we see that $ER(n)^*(\mathbb{R}P^\infty)$ may be computed from $E(n)^*(\mathbb{R}P^\infty)$ simply by base change.

[a]The very astute reader may have noticed that this argument requires commuting homology past the tensor product at each stage of the spectral sequence. That this turns out to work for us is the result of a certain property, Landweber flatness (see [**KLW16**] for more details).

1.6. The main theorem. The computation in our example generalizes to many different spaces. We have

THEOREM 2. *Let Z denote one of the Eilenberg MacLane spaces $K(\mathbb{Z}, 2q+1)$, $K(\mathbb{Z}/(2^k), 2q)$, or $K(\mathbb{Z}/(2), q)$ or one of the classifying spaces BO, BSO, $BSpin$, or $BString$ (the last for $n \le 2$ only). Then $ER(n)^*(Z)$ may be computed from $E(n)^*(Z)$ by base change. That is,*

$$ER(n)^* \otimes_{\widehat{E(n)}^*} \widehat{E(n)^*}(Z) \longrightarrow ER(n)^*(Z)$$

where the subalgebra $\widehat{E(n)}^*(Z)$ is abstractly isomorphic to $E(n)^*(Z)$ after a suitable rescaling of degrees.

Each of the spaces Z above has a corresponding space Y with projective property that allows us to produce permanent cycles via the map ψ as above. The remaining ingredient is to show that these permanent cycles generate the E_1-page over the coefficients. In [**KLW16**] this is formulated as a condition on the pair (Z, Y), leading to the definition of the titular Landweber flat Real pairs. Demonstrating the additional properties hold takes significant work, but it consists of classical nonequivariant computations that were fortunately already carried out for us in the beautiful papers [**RW80**, **RWY98**] and [**KLW04**].

References

[BW07] J. Michael Boardman and W. Stephen Wilson. $k(n)$-torsion-free H-spaces and $P(n)$-cohomology. *Canad. J. Math.*, 59(6):1154–1206, 2007.

[HK01] Po Hu and Igor Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. *Topology*, 40(2):317–399, 2001.

[JW73] David Copeland Johnson and W. Stephen Wilson. Projective dimension and Brown-Peterson homology. *Topology*, 12:327–353, 1973.

[KLW04] Nitu Kitchloo, Gerd Laures, and W. Stephen Wilson. The Morava K-theory of spaces related to BO. *Adv. Math.*, 189(1):192–236, 2004.

[KLW16] Nitu Kitchloo, Vitaly Lorman, and W. Stephen Wilson. Landweber flat real pairs and $ER(n)$-cohomology. arXiv:1603.06865, 2016.

[KW07a] Nitu Kitchloo and W. Stephen Wilson. On fibrations related to real spectra. In *Proceedings of the Nishida Fest (Kinosaki 2003)*, volume 10 of *Geom. Topol. Monogr.*, pages 237–244. Geom. Topol. Publ., Coventry, 2007.

[KW07b] Nitu Kitchloo and W. Stephen Wilson. On the Hopf ring for $ER(n)$. *Topology Appl.*, 154(8):1608–1640, 2007.

[KW08] Nitu Kitchloo and W. Stephen Wilson. The second real Johnson-Wilson theory and nonimmersions of RP^n. *Homology, Homotopy Appl.*, 10(3):223–268, 2008.

[RW80] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980.

[RW77] Douglas C. Ravenel and W. Stephen Wilson. The Hopf ring for complex cobordism. *J. Pure Appl. Algebra*, 9(3):241–280, 1976/77.

[RWY98] Douglas C. Ravenel, W. Stephen Wilson, and Nobuaki Yagita. Brown-Peterson cohomology from Morava K-theory. *K-Theory*, 15(2):147–199, 1998.

[Wil73] W. Stephen Wilson. The Ω-spectrum for Brown-Peterson cohomology. I. *Comment. Math. Helv.*, 48:45–55; corrigendum, ibid. 48 (1973), 194, 1973.

[Wil75] W. Stephen Wilson. The Ω-spectrum for Brown-Peterson cohomology. II. *Amer. J. Math.*, 97:101–123, 1975.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: vlorman@math.jhu.edu