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A user's guide: Landweber flat real pairs and $ER(n)$ cohomology

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2. Metaphors and imagery

2.1. Computations in algebraic topology. As described in the first part of this user's guide, a significant aspect of algebraic topology involves computing the values of cohomology theories on various topological spaces and studying the spaces by means of these algebraic invariants. Computations often proceed by the following metaphor:

METAPHOR 1. *Slice up the given object of interest into more manageable pieces, compute those pieces, then put them back together.*

If we are computing $E^*(X)$ for some cohomology theory E and some space X, this may apply to slicing up the space X, the theory E, or both. Examples of this notion of slicing things up into pieces include CW-decompositions, Postnikov towers, or iterating self-maps and taking quotients.

The challenge often comes in the stage of putting the basic slices back together. The methods by which topologists build things up (e.g. cofiber sequences) often induce *long* exact sequences upon applying algebraic invariants. If we built something up out of an iterated sequence of cofibrations, then in cohomology we have an iterated sequence of long exact sequences, and now the challenge is to figure out exactly how they fit together. This process is encoded in the notion of a spectral sequence. Spectral sequences are notoriously messy and complicated. Rather than get bogged down in the details, we will illustrate some ways of thinking about them by means of the example most relevant to [**KLW16**].

2.2. The Bockstein spectral sequence. Recall from the first topic our two cohomology theories of interest, Johnson-Wilson theory, $E(n)$, and Real Johnson-Wilson theory, $ER(n)$, which fit into a cofiber sequence (first constructed in [**KW07**])

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x} ER(n) \xrightarrow{p} E(n)$$

where $\lambda(n)$ is a certain positive integer and the first map is multiplication by a class $x \in \pi_{\lambda(n)} ER(n)$. This sequence relates a cohomology theory which we generally know how to compute, $E(n)$, to a cohomology theory we are interested in computing, $ER(n)$. We use the above sequence to build a tower:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{x} & ER(n) & \xrightarrow{x} & ER(n) & \xrightarrow{x} & ER(n) \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \cdots & & E(n) & & E(n) & & E(n) \end{array}$$

Note that we have now omitted the suspensions from our notation and we remember that the maps pictured may change degrees. We want to compute the $ER(n)$-cohomology of a space X, so we are going to apply the functor $[X, -]$ to the above tower. We now have an exact couple, i.e. something that looks like

$$\begin{array}{ccccccccc} \cdots \longrightarrow & ER(n)^*(X) & & \xrightarrow{x} & & ER(n)^*(X) & & \xrightarrow{x} & ER(n)^*(X) \\ \swarrow & & \nwarrow{\scriptstyle \delta} & & \swarrow{\scriptstyle p} & & \nwarrow{\scriptstyle \delta} & & \swarrow{\scriptstyle p} \\ \cdots & & & E(n)^*(X) & & & & E(n)^*(X) & \end{array}$$

where δ is the connecting homomorphism.

Let us consider how this structure gives us a method for computing $ER(n)^*(X)$ from $E(n)^*(X)$. The bottom row of the above diagram consists of the identical slices $E(n)^*(X)$, which we assume are known. We claim that $ER(n)^*(X)$, the object we want to compute, can be built up from these pieces by studying how the different copies of $E(n)^*(X)$ fit together.

Rather than thinking through this abstractly, I find it helpful to follow through what happens to actual elements of $ER(n)^*(X)$ and $E(n)^*(x)$ as we run the spectral sequence. Let's describe this by means of the following two thought experiments. The first concerns how classes in $ER(n)^*(X)$ get detected by the copies of $E(n)^*(X)$. The second concerns how certain classes in $E(n)^*(X)$ detect classes in $ER(n)^*(X)$.

Thought Experiment 1. Suppose we have a class z living in the rightmost copy of $ER(n)^*(X)$ in the exact couple above. Our goal is to detect z in one of the copies of $E(n)^*(X)$. We may map z into the first $E(n)^*(X)$ along p. If $p(z)$ is nonzero, then we have detected z and we are done. If $p(z) = 0$, then by exactness, we must have that z is divisible by x, i.e. $z = xz_1$ for some z_1. That is, z comes from the copy of $ER(n)^*(X)$ second from the right and its lift in that copy of $ER(n)^*(X)$ is given by z_1. Now we play the same game with z_1. We map it into the second copy of $E(n)^*(X)$ along p. If its image is nonzero, then we have detected it in something we know. If the image is zero, then z_1 is divisible by x (so z was a multiple of x^2) and we lift z_1 to a z_2 in the third copy of $ER(n)^*(X)$. The claim is that eventually, this process ends and we see z detected in some copy of $ER(n)^*(X)$. Why? Because $x^{2^{n+1}-1} = 0$ so eventually the only way z can be in the image of $x^{2^{n+1}-1}$ is if $z = 0$. Thus, every element of $ER(n)^*(X)$ is detected in some copy of $E(n)^*(X)$.

This gives an argument that the things we care about, classes in $ER(n)^*(X)$, show up in the slices $E(n)^*(X)$. But there is more in our copies of $E(n)^*(X)$ than just elements of $ER(n)^*(X)$. The spectral sequence starts with the copies of $E(n)^*(X)$ and, step-by-step, filters out anything that is not in detecting something coming from $ER(n)^*(X)$. So let us do a second thought experiment to trace through how we determine whether a given class in $E(n)^*(X)$ is detecting an element of $ER(n)^*(X)$.

Thought Experiment 2. Suppose we have some $w \in E(n)^*(X)$ and we want to know whether it is in the image of $ER(n)^*(X)$. We want to study the image of w along δ because if $\delta(w) = 0$, then by exactness, w must come from $ER(n)^*(X)$. On the other hand, if $\delta(w) \neq 0$, then w does not come from $ER(n)^*(X)$ and so it is not useful to us. The difficulty is that we do not know much about $\delta(w)$ because it lives in $ER(n)^*(X)$ (the very thing we are trying to compute). But we can attempt to determine whether $\delta(w) = 0$ by mapping it into the copies of $E(n)^*(X)$. Let's begin by mapping $\delta(w)$ into the next copy of $E(n)^*(X)$ to the left along p. This composite, $p \circ \delta$, is the first differential, d_1. If $d_1(w) \neq 0$, then $\delta(w) \neq 0$ and we are done with our analysis (w is not useful to us and we throw it away by taking homology with respect to d_1 as w will fail to be in the kernel and will disappear). If $d_1(w) = 0$, however, then $\delta(w)$ has a chance of being zero and we continue the analysis. Our class w has survived to the E_2-page! Now $\delta(w) \in ER(n)^*(X)$ maps to zero in $E(n)^*(X)$ under p, so it must lift to the left to the next copy of $ER(n)^*(X)$. So we lift it and map down to the next copy of $E(n)^*(X)$ along p. We are now two copies of $E(n)^*(X)$ to the left of where we started. This is the definition of d_2. Again, if we get something nonzero, then w does not show up in homology with respect to d_2 and disappears on the E_3-page. Otherwise, if $d_2(w) = 0$, then w survives to the E_3-page and it may be lifted further up the tower. Eventually, if w survives all 2^{n+1} of the differentials, then $\delta(w)$ must be in the image of $x^{2^{n+1}-1} = 0$. Then $\delta(w) = 0$, which means w is in the image of the map from $ER(n)^*(X)$. That is, w detects some class z in $ER(n)^*(X)$.

I find running these two thought experiments to be the easiest way of explaining the inner workings of the Bockstein spectral sequence. But when it comes to computing it in practice, it is useful to have effective ways of actually organizing the data. To that end, we have the following metaphor:

Metaphor 2. *A spectral sequence is a book, each page of which is a grid. Computing a differential means we can flip the page. The goal is to compute the grid (or some part of the grid) on all of the pages, and the last page (if there is one) gives us the answer we want.*

In the case of the Bockstein spectral sequence, the first page just consists of copies of $E(n)^*(X)$ as the vertical lines of the grid. Since the rth copy is going to detect x^r-torsion in $ER(n)^*(X)$, we put in a formal variable, also called x, and index the vertical copies of $E(n)^*(X)$ by powers of x. To get from each page to the next, we compute a differential as described above. Each differential forces us to reckon with the next level of torsion. Since $x^{2^{n+1}-1} = 0$, the final differential is $d_{2^{n+1}-1}$ and $E^{*,*}_{2^{n+1}}$ is the last page of the book. From the last page, we may compute $ER(n)^*(X)$, as each vertical line of the last page represents the elements of the corresponding copy of $E(n)^*(X)$ which survived and represent x^r-torsion.[1]

In practice, it is hard to compute the entire spectral sequence from scratch starting with just the first page. More typically, we need to input various pieces of outside information. One way this can happen is if we know some *permanent cycles*. This means that some part of each page of the book is known to us through the entire book, and we can hopefully fill in the rest of each page from this data. This is the idea behind the hat construction described in Topic 1.

2.3. What does it mean to know a cohomology ring? If everything works out, the end result of a spectral sequence computation is knowledge of some algebraic object. But this turns out to be a surprisingly slippery issue in the context of [**KLW16**] and many other computations in algebraic topology. The difficulty concerns what it means to 'know' a cohomology ring. One may take that to mean knowing the generators and relations. But in the case of the spaces considered in [**KLW16**], we are dealing with quotients of power series rings. The relations between the generators are power series, which have infinitely many coefficients that must be 'known'. So what is the right way to think about these objects?

One sort of answer that occurs frequently in algebraic topology is a concise description of the power series which describe the relations between the generators. For example, the $E(n)$-cohomology of $\mathbb{RP}^\infty$ described in Topic 1 involved quotienting by the 2-series, $[2]_F(u)$. This power series is determined by the formal group law F for Johnson-Wilson theory. As we further unpack this, it becomes clear that the sense in which we know the 2-series is that we have an algorithm for computing as many terms of it as we like, even though we do not have a

[1]Note: the rth vertical line of the last page of the spectral sequence is actually giving us x^r-torsion modulo x^{r+1}-torsion, so the last step is to reconstruct $ER(n)^*(X)$ from this associated graded object. Weird things can happen (for example, we could theoretically have two x^3-torsion classes multiply together to give us an x-torsion class!) so there is an art to this final step (called solving extension problems).

closed form description for its coefficients. This is the sort of knowledge we have of the $E(n)$-cohomology of all of the spaces of interest in [**KLW16**].

Note that this is not the only possibly answer one could desire[2]. But these problems are just as relevant to $E(n)$-cohomology as they are to $ER(n)$-cohomology. In fact, the main result of [**KLW16**] sidesteps this issue entirely by describing $ER(n)$-cohomology *in terms of* $E(n)$-cohomology for the spaces Z of interest:

$$ER(n)^*(Z) = ER(n)^* \otimes_{\widehat{E(n)}^*} \widehat{E(n)^*}(Z)$$

Thus, whatever the sense in which the $E(n)$-cohomology of the spaces of interest in [**KLW16**] has been computed, we compute the $ER(n)$-cohomology in the same sense.

References

[KLW16] Nitu Kitchloo, Vitaly Lorman, and W. Stephen Wilson. Landweber flat real pairs and $ER(n)$-cohomology. arXiv:1603.06865, 2016.

[KW07] Nitu Kitchloo and W. Stephen Wilson. On fibrations related to real spectra. In *Proceedings of the Nishida Fest (Kinosaki 2003)*, volume 10 of *Geom. Topol. Monogr.*, pages 237–244. Geom. Topol. Publ., Coventry, 2007.

[Su06] Hsin-hao Su. The $E(1,2)$ cohomology of the Eilenberg-MacLane space $K(\mathbb{Z},3)$. http://www.math.jhu.edu/webarchive/grad/Dissertation.Su.pdf, 2006.

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[2]An example of this is given in [**Su06**]. Let $E(1,2) = E(2)/2$. It turns out that the $E(1,2)$-cohomology of $K(\mathbb{Z},3)$ is complete with respect to the topology given by powers of v_1 even though at no point did we apply v_1-completion. This is surprising and is not easily unpacked from the description of $E(1,2)^*(K(\mathbb{Z},3))$ as a quotient of a power series ring by an ideal whose generators may be computed to arbitrary degree by an algorithm.