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A user's guide: Landweber flat real pairs and $ER(n)$ cohomology

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3. Story of the development

I will begin by outlining the history of Real Johnson-Wilson theory, $ER(n)$, before discussing how Nitu Kitchloo, W Stephen Wilson, and I came to be interested in the questions addressed in [**KLW16a**]. I will then give the specific history of [**KLW16a**], focusing on an approach that did not work, an approach that did, and what I learned about $ER(n)$ and computations from working on this project.

3.1. A brief history of $ER(n)$. Complex-oriented cohomology theories sometimes support C_2-actions, and this equivariance leads to both interesting structure and enhanced computational power. This observation traces back to Atiyah's work on Real K-theory [**Ati66**] and Araki, Fujii, Landweber, and Murayama's work on Real cobordism, $\mathbb{MU}$ [**Lan68**, **Fuj67**, **AM78**, **Ara79**]. In 2001, Hu and Kriz took the work of Araki and Landweber further by defining Real (meaning C_2-equivariant) analogs of BP, $E(n)$, and $K(n)$. Building on [**HK01**], Kitchloo and Wilson observed that the structure of the coefficients of $ER(n)$ makes it particularly amenable to computations [**KW07a**]. In the case of $ER(2)$, they used these computations in 2008 to prove many families of new non-immersion results for real projective spaces [**KW08a**, **KW08b**].

In 2009, Hill, Hopkins, and Ravenel demonstrated how using even more of the equivariance of complex-oriented theories (by studying the action of a larger group) yields deep new insights into both stable homotopy theory and geometry in their solution to the Kervaire invariant problem [**HHR09**]. Around the same time, $TMF_0(3)$, an equivariant cohomology theory built from modular forms and seemingly related to $ER(2)$ popped up in the work of Davis, Rezk, and Mahowald [**MR09**,**DM10**]. By the time I started working with $ER(n)$ in my second year of graduate school in 2011, equivariant perspectives on cohomology theories constructed from Real cobordism were developing from several directions.

3.2. How we came to this problem. The first research problem I attempted in graduate school was to determine which elements in the stable homotopy groups of spheres are detected in the coefficients of Real Johnson-Wilson theory. This seemed like a promising goal as $ER(1) = KO_{(2)}$ sees the class η and $ER(2)$ sees even more via the classes η, ν, and $\overline{\kappa}$. I hoped to find further spherical classes in the coefficients for $ER(n)$ for $n > 2$. After trying several different approaches to this problem, one of which seemed fruitful but petered out, my advisor Nitu Kitchloo and I began discussing a new approach via an $ER(n)$-based Adams-Novikov spectral sequence. The main obstruction to this program was knowing $ER(n)^*(ER(n))$, and we talked about computing $ER(n)^*(E(n))$ as an intermediate step. In one of these conversations, he pointed pointed out that even the $ER(n)$-cohomology of $\mathbb{C}P^\infty$ was not known and that it would be of independent interest. This became my thesis problem, published as [**Lor16**]. Sometime after I had worked out an approximate answer for $ER(n)^*(\mathbb{C}P^\infty)$ (the E_∞-page of the Bockstein spectral sequence), Nitu Kitchloo, W Stephen Wilson, and I began talking about extending this computation to other spaces.

The three of us first corresponded about the $ER(2)$-cohomology of $BU(q)$, but we quickly started talking more and more about approaching Eilenberg MacLane spaces as well (other than $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, which was the subject of my thesis). The space $K(\mathbb{Z}, 3)$ was the next logical choice to approach in this direction. Since $ER(n)^*(K(\mathbb{Z}, q))$ is trivial for $q > n+1$, this was the last integral Eilenberg MacLane space to compute for the theory $ER(2)$. We expected the computation to be at least as difficult as $\mathbb{C}P^\infty$ but were hopeful that it would not be intractable. The big surprise turned out to be that it is actually easier to compute! And more generally, it is easier to compute the $ER(n)$-cohomology of $K(\mathbb{Z}, \text{odd})$ than of $K(\mathbb{Z}, \text{even})$. As I will describe in the next section, it is entirely possible that expecting the computation to be difficult made us work through a lot of messy stuff that ended up being irrelevant. The previous year, Kitchloo and Wilson had computed the $ER(n)$-cohomology of BO using techniques that we generalize in [**KLW16a**]. The space BO rather than $\mathbb{C}P^\infty$ turned out to be the right analogy to focus on in our work on the $ER(n)$-cohomology of $K(\mathbb{Z}, 3)$.

3.3. How the results came about. Much of our early work on this problem happened over email, so I have a fairly extensive record of our progress. It is interesting to revisit the formation of our key ideas. In one of our first emails about $K(\mathbb{Z}, 3)$, we discussed how to think about $E(2)^*(K(\mathbb{Z}, 3))$ (the input for the BSS computing $ER(2)^*(K(\mathbb{Z}, 3))$) in terms of the sequence

$$K(\mathbb{Z}, 3) \longrightarrow BU\langle 6\rangle \longrightarrow BU\langle 4\rangle.$$

The pair $(K(\mathbb{Z}, 3), BU\langle 6\rangle)$ became our first new example of a Landweber flat real pair, and the above sequence of spaces is the first example that satisfies the conditions laid out in Section 5 of [**KLW16a**]. Later in the email chain, we briefly discussed the possibility of producing classes in the equivariant $E(2)$-cohomology of $BU\langle 6\rangle$ and then mapping them into the $ER(2)$-cohomology of $K(\mathbb{Z}, 3)$ to produce permanent cycles. This is exactly the method we ended up using in the paper. The ingredients we were missing early on were knowledge of the equivariant $E(2)$-cohomology of $BU\langle 6\rangle$ as well as the fact that the Bockstein spectral sequence could be written as a tensor product of the spectral sequence for the coefficients and an algebra of permanent cycles. The first of these two facts came from the projective property, which was a significant player in previous papers by Kitchloo and Wilson (beginning with [**KW07b**]). The second came from some remarkable left exact sequences of Hopf algebras from some great computational papers by Kitchloo, Wilson, and coauthors: [**RWY98**, **KLW04**].

Before we came around to the key ideas described in Topic 1, we spent much of the spring fiddling around with very concrete computations, especially involving the first differential in the Bockstein spectral sequence. Our hope was that the first differential would be all that we really needed to compute (as was the case, more or less, with $\mathbb{C}P^\infty$). Eventually, we started to converge on the fact that the value of the first differential on the 'hatted' power series generator of interest in $E(2)^*(K(\mathbb{Z},3))$ is zero. Around the same time, we had developed a long email chain in which we returned to the idea of using classes in the equivariant $E(2)$-cohomology of $BU\langle 6\rangle$ to produce permanent cycles. I remember a meeting in which we first recognized the argument that we had been circling around earlier of using the projective property to compute $\mathbb{E}(2)^\star(BU\langle 6\rangle)$. We also realized that this argument, together with sequences of the form

$$K(\mathbb{Z},2m+1) \longrightarrow \underline{BP\langle 2m-1\rangle}_{2(2^{2m}-1)} \longrightarrow \underline{BP\langle 2m-1\rangle}_{2^{2m}}$$

would allow us to generalize the result to all odd Eilenberg MacLane spaces. We had expected the computations for higher Eilenberg MacLane spaces to be more difficult, so this was an exciting development!

From here, the paper progressed quickly. That fall, motivated by a paper by Laures and Olbermann [**LO16**] in which they compute the $K(2)$-local $TMF_0(3)$ cohomology of $BO\langle 8\rangle$ (using techniques from [**KW14**]), we extended our results to some connective covers of BO. It was interesting to see the definition of Landweber flat real pairs apply to many more spaces than the ones that motivated it.

3.4. Lessons learned. As mentioned above, the computations in [**KLW16a**] turned out to be extremely clean. Much work would have been saved if we had known this ahead of time. As I see things now, there are two general sorts of methods to these $ER(n)$-computations: the nitty gritty approach, and the big picture approach. The nitty gritty approach consists of battling to compute each differential (often after filtering things first to simplify). The big picture approach consists of finding as many permanent cycles as possible (using some clever cofiber sequences or some geometry) and trying to generate a large part of the spectral sequence with them. The results of [**KLW16a**] turned out to be most susceptible to the latter. However, in parallel to [**KLW16a**], we also completed the computation of the $ER(2)$-cohomology of truncated projective spaces in [**KLW16b**]. This ended up being solved almost entirely by the nitty gritty method. I have come to see that both approaches are useful to making $ER(n)$ computations, and this fact is essential to its charm. There is just enough structure that one can compute and just enough chaos to make the results of the computations new and exciting.

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