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A user's guide: Landweber flat real pairs and $ER(n)$ cohomology

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4. Colloquial summary

The work carried out in [**KLW16a**] is part of a broad subfield of math called *algebraic topology*. At its essence, algebraic topology studies the rich and mysterious world of topology (a certain kind of geometry) by connecting it to the more tractable world of algebra. Before we discuss the work of interest to the rest of this user's guide, let's unpack this statement.

4.1. Geometry is more than what we see. The sorts of shapes algebraic topologists study include any that you may care about. Some of them you see in the world around you: spheres, donuts, knots, or more complicated shapes like carburetors. But they also, importantly, include high dimensional shapes that we do not see with our eyes but which we can nevertheless describe abstractly.

QUESTION 4.1. *If we cannot see these shapes in the world around us, why should we care about them?*

The answer is that from a topologist's point of view, having a notion of 'nearness' is much more important than actually being able to see (either physically or in the mind's eye) the shapes of interest. Here are some examples of important high-dimensional geometric objects that never enter our visual field:

(a) the shape of all patient records of a given hospital[1],
(b) the shape of the voting preferences of all citizens in a given place,
(c) the shape of all configurations of n particles in space[2],
(d) the shape of all configurations of a robot with three arms of two joints each,

[1] The dimension is given by the number of parameters (e.g. age, height, weight, blood sugar) we keep for each patient.

[2] Example: the classical three-body problem takes place in 18-dimensional space. For each of three bodies, we need three dimensions to specify a position and three dimensions to specify a velocity.

(e) the shape of all possible ways of packing bowling balls in a barrel,
(f) generally, the shape of all possible configurations of a system with quantifiable parameters.

For each of these examples we have some notion of what it means for two things to be near or far from each other. For example, we can talk about configuration A of a robot's arms being near configuration B if, overall, the arm placements in configuration A are near the arm placements in configuration B. This is, roughly, why we are justified in calling each of the above examples shapes even though we do not see them with our eyes.

The field of topology is specifically concerned with the *qualitative* aspects of these kinds of geometric objects. That is, a topologist is less interested in the precise distance between two points in a shape than in global features, such as whether there is a hole in the middle of the shape that separates one region from another. While interpreting what such a feature says about the actual physical situation is an important question[3], pure topology is concerned with understanding what sorts of global phenomena can occur, how they are related to each other, and how we can classify shapes based on these phenomena.

4.2. Cohomology theories. Algebraic topologists study the above sorts of shapes by coming up with various concrete and quantitative ways to measure their qualitative features. Some such ways of measuring are called *cohomology theories*, and they associate to each kind of shape a certain kind of very tractable and understood object. As a first approximation you can take our well-understood object to be a number.

Let's look at a concrete example. Suppose we have the following two shapes and we want to tell them apart.

[3]This is one aspect of the field of applied algebraic topology, which has found some striking use in the past decade to problems in fields as diverse as medicine, sensor networks, computer vision, and social choice theory, to name just a few.

One way to do this is to count the number of holes in each region and notice that we do not get the same numbers for each shape.[4]

This example illustrates that while shapes can be complicated, numbers are very concrete things. Algebraic topologists have invented many other ways of assigning numbers (or other rigid and well-understood objects like groups or vector spaces or rings) to geometric shapes that measure their various features. The results of [**KLW16a**] are concerned with one specific cohomology theory: *Real Johnson-Wilson theory.*

4.3. Real Johnson-Wilson theory and torsion. Real Johnson-Wilson theory has two advantages compared to other cohomology theories. The first is that the numbers it assigns shapes often exhibit a very special algebraic phenomenon called *torsion.* To get a sense of what torsion means, imagine the hours on a clock. The hour of day is a number but unlike the ordinary number line which keeps going indefinitely, once we reach the end of the (half) day at 12 hours, the hours start over. Torsion is the phenomenon of adding something to itself some number of times and getting back to the same place. An ordinary 12-hour clock exhibits 12-torsion. The numbers[5] produced by Real Johnson-Wilson theory are 2-torsion.

Many of the least understood phenomena in topology are closely connected with torsion. In fact, perhaps the biggest motivating problem in algebraic topology is computing the homotopy groups of spheres, which are algebraic objects that, roughly, contain all the information about all possible ways of constructing all possible shapes. The homotopy groups of spheres are almost entirely torsion, and they are in large part unknown.

The computations of [**KLW16a**] and research with Real Johnson-Wilson theory in general work because they detect only a small bit of torsion at a time, and they are constructed from a certain torsion-free cohomology theory (Johnson-Wilson theory[6]). Being torsion-free, computations with Johnson-Wilson theory are often susceptible to one of the most powerful tools in the mathematician's general toolbox: linear algebra. They are well-understood, and the results of [**KLW16a**] extend many known computations to the 'Real' context where there is interesting torsion to reckon with.

[4]This example has much more complex analogs. Imagine the surface of a sphere. Now imagine the surface of a donut. These also have holes (i.e. the sphere has a 2-dimensional empty space inside of it), but of different natures from our shapes above. In higher dimensions, this gets even more complicated and the phenomena that occur become much more diverse and much stranger.

[5]more properly, groups

[6]Johnson-Wilson theory and Real Johnson-Wilson theory are *very different* cohomology theories. Real Johnson-Wilson theory should not be thought of as a version of Johnson-Wilson theory. Rather, it is constructed from Johnson-Wilson theory in a way that transmutes its structure in a controlled but significant way–in particular, it introduces torsion.

The computations in [**KLW16a**] are part of a larger program to study the way in which the torsion in Real Johnson-Wilson theory distorts some of the classical structure present in Johnson-Wilson theory. Besides being interesting for their own sake, these results sometimes yield applications to concrete geometric questions[7]. This is an important direction of future research. Our hope is that the results of [**KLW16a**] (as well as [**Lor16**, **KLW16b**] and work in progress) can be leveraged to produce some new geometric applications. In other words, now that we know more about the structure of Real Johnson-Wilson theory, we want to know what this tells us about the shapes that we care about.

References

[KLW16a] Nitu Kitchloo, Vitaly Lorman, and W. Stephen Wilson. Landweber flat real pairs and $ER(n)$-cohomology. arXiv:1603.06865, 2016.

[KLW16b] Nitu Kitchloo, Vitaly Lorman, and W. Stephen Wilson. The $ER(2)$-cohomology of $B\mathbb{Z}/(2^q)$ and $\mathbb{C}P^n$. arXiv:1605.07401, 2016.

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[7]For example nonimmersion problems, which have to do with when we can squeeze certain kinds of high-dimensional shapes into a high-dimensional Euclidean space.