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A user's guide: An equivariant tensor product on Mackey functors

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1. Key insights and central organizing principles

1.1. Mackey functors are the new groups. Let G be a finite group. A goal of equivariant stable homotopy theory is to understand the homotopy groups of genuine G-spectra. Equivariant spectra are like spaces with a lot of extra structure. Hence, their homotopy "groups" also have a lot of extra structure. This extra structure stems from the fact that we want to define the k^{th}-stable homotopy group $\underline{\pi}_k$ of a G-spectrum X in such a way that it reflects not only the action of G on X but also the actions of all subgroups of G on X. So, $\underline{\pi}_k(X)$ is not merely a group. Instead, we define $\underline{\pi}_k(X)$ by first considering the group of homotopy classes of H-equivariant maps from the k^{th} sphere spectrum into X for all subgroups H of G. Then as H varies these abelian groups fit together to form a *Mackey functor*. We define $\underline{\pi}_k(X)$ to be this Mackey functor.

If X has *even more* structure, in particular, if X is a commutative G-ring spectrum, then in [**Bru07**] Brun shows that its zeroeth stable homotopy group is more than just a Mackey functor. A commutative G-ring spectrum comes equipped with norm maps, and we see the algebraic shadows of these norm maps in $\underline{\pi}_0(X)$. Hence, $\underline{\pi}_0(X)$ is a Mackey functor with commutative ring properties *and* internal norm maps that are like multiplicative transfer maps. It is a *Tambara functor.*

We want there to be a nice symmetry between G-spectra and their homotopy groups. So, we want the following idea to be true.

KEY IDEA 1.1. *Since Mackey functors are the new groups, Tambara functors should be the new rings.*

In other words, we want Tambara functors to play the role of rings in the category of Mackey functors. The category of G-Mackey functors $\mathcal{M}ack_G$ is a symmetric monoidal category, and so we can use the symmetric monoidal product and classic category theory to define ring objects (i.e. monoids with respect to the monoidal product) in $\mathcal{M}ack_G$. But alas! The ring objects under the symmetric monoidal product are merely Green functors. They are *not* Tambara functors. Green functors have less structure than Tambara functors because they do not have norm maps. Therefore, the goal of [**Maz16**] is as follows.

GOAL 1.2. *For G a cyclic p-group, develop an equivariant symmetric monoidal structure on $\mathcal{M}ack_G$ under which Tambara functors are the commutative ring objects.*

Hill and Hopkins developed an appropriate notion of equivariant symmetric monoidal, calling it *G-symmetric monoidal*. They call the ring objects under this structure *G-commutative monoids* [**HH13**]. Hill and Hopkins [**HH13**] and Hoyer [**Hoy14**] have independently defined G-symmetric monoidal structures on $\mathcal{M}ack_G$. Ullman [**Ull13**] then provides an algebraic description of Hill and Hopkins' structure. At the core of these structures are functors $N_H^G\colon \mathcal{M}ack_H \to \mathcal{M}ack_G$ for all subgroups H of G that send an H-Mackey functor to a G-Mackey functor. Hill and Hopkins and Ullman defined N_H^G by passing to an H-spectrum via the Eilenberg-MacLane functor, applying the Hill-Hopkins-Ravenel norm functor and then returning to G-Mackey functors via $\underline{\pi}_0$. Hoyer defined the functors N_H^G via coends.

In [**Maz16**], for G a cyclic p-group, we present a novel approach to building a G-symmetric monoidal structure on $\mathcal{M}ack_G$. Our structure does not involve spectra or coends. Instead, we develop a *concrete* G-symmetric monoidal structure by constructing new functors N_H^G using only the algebraic properties of Mackey functors and Tambara functors.

Hoyer shows that all of these G-symmetric monoidal structures are isomorphic [**Hoy14**]. However, there are still advantages and disadvantages to each. Hill and Hopkins', Ullman's and Hoyer's definitions of the functors N_H^G are easier to state, and it is easier to use these definitions to further develop concepts in equivariant stable homotopy theory. However, it is difficult to see how these functors affect any given Mackey functor. Furthermore, in order to really understand these definitions you need a good understanding of spectra. On the other hand, the definition of N_H^G in [**Maz16**] is nice because even though it is a bit messy, it does not rely on extensive knowledge of spectra or category theory, and it gives us a clearer understanding of what N_H^G does to individual Mackey functors. Thus, it is accessible to a wider range of mathematicians.

1.2. Discussion of the main construction. From here on let G be a cyclic p-group. Further, let $\mathscr{S}et_G^{Fin}$ be the category of finite G-sets. A G-symmetric monoidal structure on $\mathcal{M}ack_G$ is a functor

$$(-)\otimes(-)\colon \mathscr{S}et_G^{Fin} \times \mathcal{M}ack_G \to \mathcal{M}ack_G$$

that satisfies the properties given in Definition 5.1 of [**Maz16**]. In particular, $(-)\otimes(-)$ must break up disjoint unions of G-sets over the symmetric monoidal product $\Box$ in $\mathcal{M}ack_G$. That is, given G-sets X and Y and G-Mackey functor $\underline{M}$,

$$(X \amalg Y)\otimes \underline{M} = (X\otimes \underline{M}) \Box (Y \otimes \underline{M}).$$

Since every G-set can be written as a disjoint union of orbits, we can define $(-) \otimes (-)$ by defining $G/H \otimes \underline{M}$ for all orbits G/H and all G-Mackey functors $\underline{M}$. Hence, let $i_H^* \colon \mathcal{M}ack_G \to \mathcal{M}ack_H$ be the forgetful functor that sends a G-Mackey functor to its underlying H-Mackey functor. The following key idea completely determines our G-symmetric monoidal structure.

Key Idea 1.3. *For all G-Mackey functors $\underline{M}$ and all orbits G/H we define $G/H \otimes \underline{M}$ to be the composition $N_H^G i_H^* \underline{M}$.*

Thus, we define the G-Mackey functor $G/H \otimes \underline{M}$ by first reducing $\underline{M}$ down to an H-Mackey functor using i_H^* and then building a new G-Mackey functor via the functor $N_H^G \colon \mathcal{M}ack_H \to \mathcal{M}ack_G$. Indeed, the bulk of [**Maz16**] focuses on constructing the functors N_H^G for all subgroups H of G. The proof of Theorem 1.1 is not monumental because we meticulously developed N_H^G in Sections 3 and 4. Further, we define N_H^G so that our definition of the functor $(-)\otimes(-)$ maintains all of the properties of a G-symmetric monoidal structure and so that Tambara functors are the ring objects under this structure.

Key Idea 1.4. *All of the hard work lies in building the functors*

$$N_H^G \colon \mathcal{M}ack_H \to \mathcal{M}ack_G.$$

Let H be a subgroup of G, and let $\underline{M}$ be an H-Mackey functor. To define the functor N_H^G we first construct the G-Mackey functor $N_H^G \underline{M}$. We use the word "construct" deliberately. Definitions 3.2, 3.3, 3.4 and 3.9 of [**Maz16**] give detailed instructions for building each piece of $N_H^G \underline{M}$. We explicitly define every module $(N_H^G \underline{M})(G/K)$ and all restriction and transfer maps for all $K \leq G$. The collection of these definitions is long and looks intimidating, but it is actually fairly simple in the sense that it is just a detailed list of generators and relations. We then prove that the map $\underline{M} \mapsto N_H^G \underline{M}$ is a symmetric monoidal functor from $\mathcal{M}ack_H$ to $\mathcal{M}ack_G$ in Theorems 4.1 and 4.11.

So, how do we actually build the G-Mackey functor $N_H^G \underline{M}$? The construction is motivated by the fact that at the end of the day we want to use the collection of functors N_H^G for all subgroups H of G to create Tambara functors.[1]

Key Idea 1.5. *The G-Mackey functor $N_H^G \underline{M}$ must encode properties of a Tambara functor (especially the internal norm map N_H^G and Tambara reciprocity).*

We build $N_H^G \underline{M}$ from the ground up using the fact that all subgroups of G are nested. So, this is where we need the fact that G is a cyclic p-group C_{p^n}. We first define $(N_H^G \underline{M})(G/e)$. Then we use that module to define $(N_H^G \underline{M})(G/C_p)$ and continue up the ladder until we define $(N_H^G \underline{M})(G/G)$. We define the transfer and restriction maps so that they naturally satisfy all necessary properties of the

[1] In Topic 2 we give more details and examples of $N_H^G \underline{M}$. We also give examples of ladder diagrams of Mackey functors and Tambara functors. If you have not yet seen these diagrams, it may be helpful to read Topic 2 before finishing Topic 1.

corresponding maps of a Mackey functor. Moreover, the restriction maps satisfy properties analogous to the inherent properties of restriction maps in Tambara functors.

Further, to define $N_H^G\underline{M}$ we split this Mackey functor at the module $(N_H^G\underline{M})(G/H)$ and think about the chunk of $N_H^G\underline{M}$ that is below $(N_H^G\underline{M})(G/H)$ (i.e. $(N_H^G\underline{M})(G/H')$ for all subgroups H' of H) separately from the chunk above $(N_H^G\underline{M})(G/H)$.

The chunk below $(N_H^G\underline{M})(G/H)$:

If H' is a subgroup of H, then we define $N_H^G\underline{M}(G/H')$ to be $\underline{M}^{\Box|G/H|}(H/H')$. Further, the symmetric monoidal product is like a tensor product, and we can use this idea to get a better picture of what $\underline{M}^{\Box|G/H|}(H/H')$ looks like. Ideally, we would like $\underline{M}^{\Box|G/H|}(H/H')$ to simply be $\underline{M}(H/H')^{\otimes|G/H|}$. However, this module is not big enough to allow us to define the transfer map into $\underline{M}^{\Box|G/H|}(H/H')$ in a compatible way. Thus, letting H'' be the maximal subgroup of H', we define $\underline{M}^{\Box|G/H|}(H/H')$ to be a direct sum of $\underline{M}(H/H')^{\otimes|G/H|}$ and a copy of the module $(\underline{M}^{\Box|G/H|})(H/H'')$ modulo a Weyl action. We then define the transfer map $tr_{H''}^{H'}$ to be the canonical quotient map onto the latter summand. The two summands are connected by quotienting out by a submodule that creates relations analogous to Frobenius reciprocity. Hence, at the H' level, $N_H^G\underline{M}$ looks like the picture below.

$$(N_H^G\underline{M})(G/H'): \qquad \underline{M}^{\Box|G/H|}(H/H') = (\text{Tensor Part} \oplus \overbrace{\text{Image of Transfer Map}}^{(\underline{M}^{\Box|G/H|})(H/H'')/_{W_{H'}(H'')}})/_{FR}$$

$$res_{H''}^{H'} \downarrow \quad \uparrow tr_{H''}^{H'}$$

$$(N_H^G\underline{M})(G/H''): \qquad \underline{M}^{\Box|G/H|}(H/H'')$$

The chunk above $(N_H^G\underline{M})(G/H)$:

If K is a subgroup such that $H < K \leq G$, then $(N_H^G\underline{M})(G/K)$ looks a lot scarier, but its general idea is the same. We want $(N_H^G\underline{M})(G/K)$ to feel like a part of a Tambara functor, but we need to create space for the image of the transfer map. So, this module is a direct sum involving a free summand generated by copies of $\underline{M}(H/H)$ and a transfer summand. The transfer summand is completely analogous to the image of the transfer map in the above diagram. The two summands are once again connected by quotienting out by the right submodule. However, this time we want the submodule to create relations like Tambara reciprocity. We will then be able to extract properties of a Tambara functor from the free summand. Below we showcase the K level of $N_H^G\underline{M}$ where

K is the subgroup in which H is maximal.

$$\begin{array}{ll} (N_H^G\underline{M})(G/K)\colon & \Big(\text{Free Part} \oplus \overbrace{\text{Image of Transfer Map}}^{(N_H^G\underline{M})(G/H)/_{W_K(H)}}\Big)/_{TR} \\ & \quad res_H^K \downarrow \uparrow tr_H^K \\ (N_H^G\underline{M})(G/H)\colon & \underline{M}^{\Box|G/H|}(H/H) \end{array}$$

In Section 4 of [**Maz16**] we develop all of the necessary properties of the functors N_H^G so that we can use them to create a G-symmetric monoidal structure on $\mathcal{M}ack_G$ under which Tambara functors are the G-commutative monoids.

Finally, in Section 5 we prove that our definition of $(-) \otimes (-)$ is in fact a G-symmetric monoidal structure on $\mathcal{M}ack_G$ and prove that Tambara functors are the G-commutative monoids. Most aspects of these two proofs are straightforward because we so carefully constructed the functors N_H^G. However, one of my favorite parts of this paper is showing that a G-commutative monoid is a G-Tambara functor. Indeed, if a Mackey functor $\underline{M}$ is a G-commutative monoid, then maps $X \to Y$ of G-sets induce maps $X \otimes \underline{M} \to Y \otimes \underline{M}$ of Mackey functors. To show that $\underline{M}$ is a Tambara functor we first show that $\underline{M}$ is a Green functor (i.e. a Tambara functor without the internal norm maps). This proof involves chasing diagrams of disjoint unions of single point sets. Then given a subgroup H of G we define the internal norm map $N_H^G\colon \underline{M}(G/H) \to \underline{M}(G/G)$ by creating a composition

$$\underline{M}(G/H) \to N_H^G i_H^* \underline{M}(G/G) \to \underline{M}(G/G).$$

The first map of the composition follows from the fact that $\underline{M}(G/H)$ is isomorphic to $i_H^*\underline{M}(H/H)$ and so nicely maps into the free part of $N_H^G i_H^* \underline{M}(G/G)$. The second is part of the collection of homomorphisms that make up the morphism of Mackey functors $G/H \otimes \underline{M} \to G/G \otimes \underline{M}$. This composition possesses all properties of the internal norm map of a Tambara functor because of the way we defined $(N_H^G i_H^* \underline{M})(G/G)$.

Key Idea 1.6. *Under the G-symmetric monoidal structure that we created Tambara functors are the G-commutative monoids.*

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