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A user's guide: An equivariant tensor product on Mackey functors

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2. Metaphors and imagery

For a cyclic p-group G, the backbone of the G-symmetric monoidal structure that we construct in [**Maz16**] is the collection of norm functors $N_H^G \colon \mathcal{M}ack_H \to \mathcal{M}ack_G$ for all subgroups H of G. These functors build a G-Mackey functor that looks like a Tambara functor using only the basic, algebraic properties of Mackey functors and Tambara functors. More specifically, given an H-Mackey functor $\underline{M}$, we build a G-Mackey functor $N_H^G\underline{M}$ that somehow encodes the properties of the norm maps of a Tambara functor. Then we can use $N_H^G\underline{M}$ to define these norm maps by creating a composition of homomorphisms that factors through $N_H^G\underline{M}$.

When creating the Mackey functor $N_H^G\underline{M}$ my goal was to build a "ladder diagram" of a Mackey functor that looks and acts like a Tambara functor. I spent a lot of time building and playing with these diagrams, especially for C_4-Mackey and Tambara functors. Moreover, I used the fixed point Tambara functor to help build intuition for the construction of $N_H^G\underline{M}$. Thus, here we will focus on constructing such ladder diagrams. We will specifically build diagrams for the fixed point C_4-Mackey functor and the fixed point C_4-Tambara functor. Finally, the general definition of the Mackey functor $N_H^G\underline{M}$ can be overwhelming. Thus, to demonstrate how to think about it we will build a ladder diagram for the C_4-Mackey functor $N_{C_2}^{C_4}\underline{M}$.

2.1. Mackey functors. Throughout this discussion let G be a cyclic p-group. Formally, a G-Mackey functor consists of a pair of functors from the category of finite G-sets to the category of abelian groups that satisfy certain properties. But because of these properties, a G-Mackey functor $\underline{M}$ boils down to a collection of modules

$$\{\underline{M}(G/H) : H \leq G\}$$

along with maps between them. Since all subgroups of G are nested (because G is a cyclic p-group), we can imagine $\underline{M}$ as a ladder in which each rung is a module $\underline{M}(G/H)$. Then if H is a subgroup of K, there are two maps between $\underline{M}(G/H)$ and $\underline{M}(G/K)$: the *transfer* map $tr_H^K\colon \underline{M}(G/H) \to \underline{M}(G/K)$ going up the ladder and the *restriction* map $res_H^K\colon \underline{M}(G/K) \to \underline{M}(G/H)$ going down the ladder. Moreover, if K' is a subgroup of G such that $H < K' < K$, then $res_H^K = res_H^{K'} res_{K'}^K$ and $tr_H^K = tr_{K'}^K tr_H^{K'}$, so we only need to determine tr_H^K and res_H^K when H is the maximal subgroup of K. Therefore, we picture Mackey functors in diagrams like Figure 1.

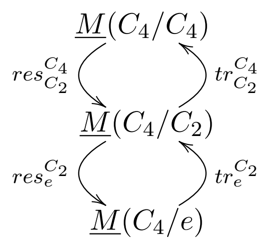
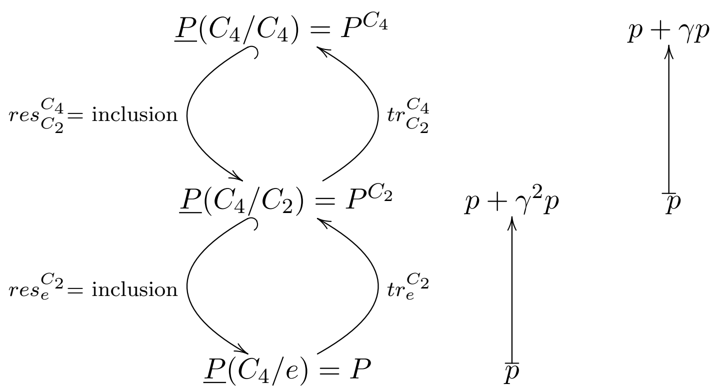


FIGURE 1. $\underline{M}$ is a C_4-Mackey Functor

Further, the restriction and transfer maps must play by certain rules. For example, the Weyl groups $W_K(H)$ act on each module $\underline{M}(G/H)$ whenever H is a subgroup of K. If γ is the generator of G and x is in $\underline{M}(G/H)$, then $res_H^K tr_H^K(x) = \sum_{\gamma^t \in W_K(H)} \gamma^t x$.

EXAMPLE 2.1. Let P be a C_4-module and let P^H denote the H fixed points of P for all subgroups H of C_4. Further, let γ be the generator of C_4. In the diagram below we have constructed the fixed point C_4-Mackey functor $\underline{P}$ from P. Notice that $tr_e^{C_4}(p) = p + \gamma p + \gamma^2 p + \gamma^3 p$ for all p in $\underline{P}(C_4/e)$.



2.2. Tambara functors. A Tambara functor is a Mackey functor with a lot of extra structure. So, to create a G-Tambara functor $\underline{S}$ we start with a Mackey functor ladder diagram and add a bunch of extra information to it. First, every $\underline{S}(G/H)$ is now a commutative ring instead of a module and the restriction maps become ring homomorphisms. (The transfer maps do not.) Then, whenever H is a subgroup of K we add another map $\underline{S}(G/H) \to \underline{S}(G/K)$ going up the ladder. This map is called a *norm* map and is denoted N_H^K. The norm maps are the multiplicative analogues of the transfer maps. For example, the norm maps are homomorphisms of multiplicative monoids (but are not additive), and $res^K_H N^K_H(x) = \prod_{\gamma^t \in W_K(H)} \gamma^t x$. A ladder diagram for a C_4-Tambara functor is given in Figure 2.

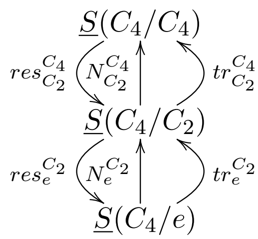


Figure 2. $\underline{S}$ is a C_4-Tambara Functor

Further, Tambara functors satisfy *Tambara reciprocity*, which tells us how the norm maps interact with sums and transfer terms. In particular, the norm maps are not additive, so, in general, $N^K_H(a+b) \neq N^K_H(a) + N^K_H(b)$, but via Tambara reciprocity, the norm maps are additive up to a transfer term. Hence,

$$N^K_H(a+b) = N^K_H(a) + N^K_H(b) + tr(-).$$

The specific make up of the transfer term $tr(-)$ depends on K and H and consists of sums of products of various Weyl conjugates of a and b. Similarly, if H' is a subgroup of H, there is a Tambara reciprocity formula that allows us to rewrite $N^K_H tr^H_{H'}(x)$ as the transfer $tr^K_{H'}$ of some element. Further, the Tambara reciprocity formulas depend only on the subgroups H', H and K of G. Every G-Tambara functor satisfies the same Tambara reciprocity properties. For example, in *every* C_{2^n}-Tambara functor,

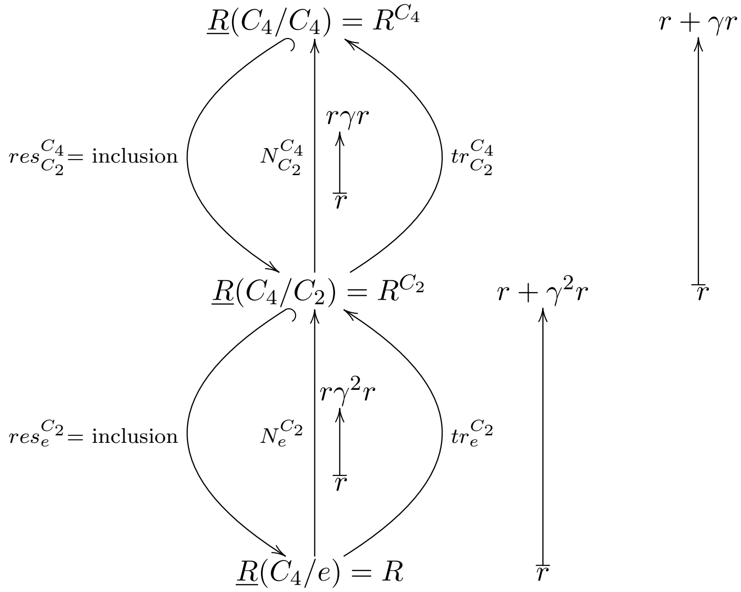
$$N^{C_2}_e(a+b) = N^{C_2}_e(a) + N^{C_2}_e(b) + tr^{C_2}_e(a\gamma^i b),$$

where γ^i is the generator of $W_{C_2}(e)$, and in *every* C_{2^n}-Tambara functor when $n \geq 2$,

$$N^{C_4}_{C_2} tr^{C_2}_e(x) = tr^{C_4}_e(x\gamma^j x),$$

where γ^j is the generator of $W_{C_4}(e)$.

Example 2.2. We will build the fixed point C_4-Tambara functor $\underline{R}$ by adding structure to the fixed point C_4-Mackey functor in Example 2.1. First, start with a commutative C_4-ring R, instead of just a C_4-module. Then each $\underline{R}(C_4/H)$ is still R^H, and the restriction and transfer maps are as given in Example 2.1. Notice that the restriction maps are ring homomorphisms, but the transfer maps are not because they are not multiplicative. Adding the norm maps to the ladder diagram results in the picture below. Lastly, note that $N^{C_4}_e(r) = r(\gamma r)(\gamma^2 r)(\gamma^3 r)$ and that we can verify that $\underline{R}$ satisfies the Tambara reciprocity formulas mentioned above.



In [**Maz16**] we develop the formulas for Tambara reciprocity by chasing exponential diagrams. But these formulas are universally determined by the group G. This means that they will be the same in every G-Tambara functor. In particular, the formula for $N^G_H(a+b)$ in the G-fixed point Tambara functor is the same as the formula for $N^G_H(a+b)$ in *any other* G-Tambara functor. So, we can cheat and use the fixed point G-Tambara functor to come up with the appropriate formulas for the norm of a sum and the norm of a transfer in *any* G-Tambara functor. For example, in the C_4-fixed point Tambara functor above,

$$N^{C_4}_e(a+b) = (a+b)\gamma(a+b)\gamma^2(a+b)\gamma^3(a+b).$$

If we expand the righthand side of this equation, we can re-write it as

$$N^{C_4}_e(a) + N^{C_4}_e(b) + tr^{C_4}_{C_2}\left(N^{C_2}_e(a\gamma b)\right) + tr^{C_4}_e(a\gamma a\gamma^2 a\gamma^3 b + b\gamma b\gamma^2 b\gamma^3 a + a\gamma b\gamma^2 b\gamma^3 a).$$

Since this formula is universally determined by C_4, it will hold in every C_4-Tambara functor.

2.3. The Mackey functor $N^{C_4}_{C_2}\underline{M}$. Now, given an H-Mackey functor $\underline{M}$, we want to use these ladder diagrams of Mackey functors and Tambara functors to understand the G-Mackey functor $N^G_H\underline{M}$. But, when we try tackle this general case we quickly become bogged down in notation and details. So to simplify the exposition we will start with a C_2-Mackey functor $\underline{M}$ and build the ladder diagram for the C_4-Mackey functor $N^{C_4}_{C_2}\underline{M}$.

Recall that $N^{C_4}_{C_2}\underline{M}$ is a Mackey functor, but we want it to feel like a Tambara functor. Thus, let $\square$ be the symmetric monoidal product (i.e. the box product) in the category of C_4-Mackey functors. We define $(N^{C_4}_{C_2}\underline{M})(C_4/e)$ to be $(\underline{M}\ \square$

$\underline{M})(C_2/e)$ and $(N_{C_2}^{C_4}\underline{M})(C_4/C_2)$ to be $(\underline{M}\Box\underline{M})(C_2/C_2)$. The maps $res_e^{C_2}$ and $tr_e^{C_2}$ are the restriction and transfer maps of the box product definition (Definition 3.1 in [**Maz16**]). It remains to define $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$. As discussed in Part 1 of this User's Guide, $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$ consists of a free part and the image of the transfer map $tr_{C_2}^{C_4}$. We think of the free part as the home for norms, and so when we use this construction to create Tambara functors, we will pull norm elements from the free part. Thus, since the norms in a Tambara functor must satisfy Tambara reciprocity, we need to mimic this property in $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$. Hence, we quotient $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$ by the *Tambara reciprocity submodule* TR, and

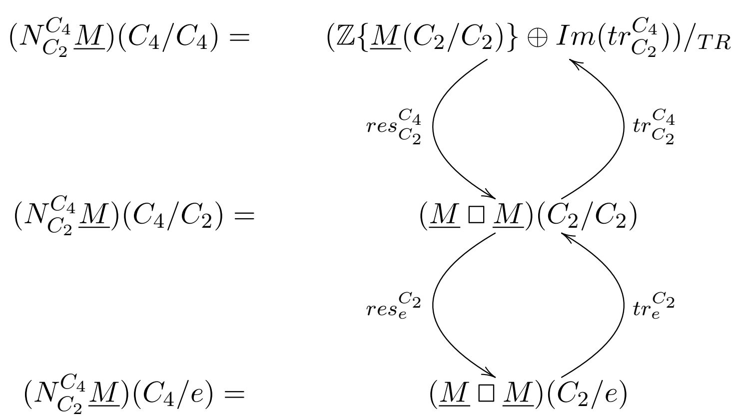
$$(N_{C_2}^{C_4}\underline{M})(C_4/C_4) = (\mathbb{Z}\{\underline{M}(C_2/C_2)\} \oplus Im(tr_e^{C_2}))/_{TR}.$$

We denote a generator of $\mathbb{Z}\{\underline{M}(C_2/C_2)\}$ by $N(a)$ for a in $\underline{M}(C_2/C_2)$, and so TR is generated by elements of the following forms for all a and b in $\underline{M}(C_2/C_2)$ and x in $\underline{M}(C_2/e)$:

$$N(a+b) - N(a) - N(b) - tr_{C_2}^{C_4}(a \otimes b)$$

$$N(tr_e^{C_2}(x)) - tr_e^{C_4}(x \otimes x).$$

These relations look familiar, right? We designed them so that they mirror the Tambara reciprocity relations that we saw in the discussion of Tambara functors. We visualize $N_{C_2}^{C_4}\underline{M}$ using the ladder diagram below.



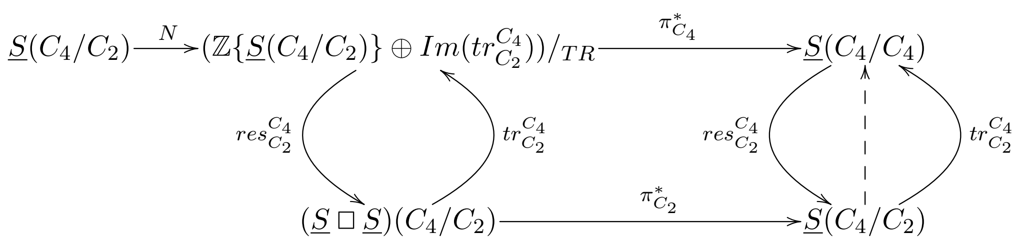
Finally, we will use this construction to define the internal norm maps of a Tambara functor. We first endow the category of G-Mackey functors with the G-symmetric monoidal structure defined in Theorem 5.2 of [**Maz16**]. So, by Proposition 5.8 of [**Maz16**] we know that if a G-Mackey functor $\underline{S}$ is a G-commutative monoid, then it has the extra structure of a Tambara functor. In particular, if $\underline{S}$ is a G-commutative monoid, then we can use the norm functors $N_H^G\colon \mathcal{M}ack_H \to \mathcal{M}ack_G$ to define the internal norm maps in $\underline{S}$. We will conclude by demonstrating how to build the internal norm map $N_{C_2}^{C_4}$ of a C_4-Tambara functor using the norm functor $N_{C_2}^{C_4}\colon \mathcal{M}ack_{C_2} \to \mathcal{M}ack_{C_4}$.

Let $\underline{S}$ be a C_4-Mackey functor that is a C_4-commutative monoid. Since $\underline{S}$ is a C_4-commutative monoid, we have a map

$$\pi^*\colon C_4/C_2 \otimes \underline{S} \to C_4/C_4 \otimes \underline{S}.$$

Recall from Part 1 of this guide that $C_4/C_2 \otimes \underline{S} = N_{C_2}^{C_4} i_{C_2}^* \underline{S}$ where $i_{C_2}^*$ is the forgetful functor, and so π^* is a morphism $N_{C_2}^{C_4} i_{C_2}^* \underline{S} \to \underline{S}$ of C_4-Mackey functors. Further, $(i_{C_2}^* \underline{S})(C_2/C_2)$ is isomorphic to $\underline{S}(C_4/C_2)$. Hence, we have a map $N\colon \underline{S}(C_4/C_2) \to (N_{C_2}^{C_4} i_{C_2}^* \underline{S})(C_4/C_4)$ that sends an element a in $\underline{S}(C_4/C_2)$ to the corresponding generator of the free summand of $(N_{C_2}^{C_4} i_{C_2}^* \underline{S})(C_4/C_4)$.

We use the ladder diagram below to define the internal norm map $N_{C_2}^{C_4}\colon \underline{S}(C_4/C_2) \to \underline{S}(C_4/C_4)$ of $\underline{S}$. This norm map is the dashed map on the right side of the diagram, and we define it to be the composition $\pi_{C_4}^* N$ that goes across the top of the diagram. This composition will satisfy all properties required of a norm map in a Tambara functor because of the way we constructed $N_{C_2}^{C_4} i_{C_2}^* \underline{S}$.



References

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