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A user's guide: An equivariant tensor product on Mackey functors

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4. Colloquial summary

In the past if a lay-person asked me about this project or my field of research, my eyes would light up and the following Taylor Swift lyric would pop into my brain.

"I can show you incredible things
Magic, madness, heaven, sin." [**Swi14**]

I would eagerly explain how my goal is to learn about fancy, mathematical objects called G-spectra. However, because G-spectra prove difficult to study directly, we assign objects called Mackey functors to them, and we subsequently learn about G-spectra by studying Mackey functors. Unfortunately, my explanation inevitably caused most eyes to quickly glaze over no matter how enthusiastically I presented it. The technical aspects of algebraic topology and my project in particular make it difficult to appreciate their beauty. For example, it took me three years of graduate school in pure mathematics and fluency in a language of mathematics called category theory to even understand the *question* I hoped to answer. This project lives in such a small niche of mathematics that I cannot easily explain it to my colleagues at Elon because they do not research algebraic topology. If my professional colleagues struggle with the language and technicalities associated with studying G-spectra and Mackey functors, then I find it harder still to try to explain it to a general audience.

However, when I mention my career choice to general audiences they are usually intrigued by my life as a math professor. While not necessarily interested in the nuts and bolts of algebraic topology, they are often curious about my personal relationship with my field of research and about the impacts of my research on society. Questions about how I ended up in this field, why I find it interesting and what will come out of my research come up frequently in social engagements, yet still catch me off guard. I never really know how to answer these questions, and that is not only a disservice to my conversation partner, my career, and myself but also to the entire mathematical community. Good answers to such questions will help a general audience appreciate mathematics and mathematics research even when the research itself is too technical to follow. Thus, I have recently spent time pondering these questions and reflecting on why I chose a career in mathematics.

Therefore, I have divided this summary into two parts. The first is a very brief and general explanation of this project. It is aimed at an undergraduate student who has taken abstract algebra. My goal is to create a summary that is helpful for an undergraduate who is interested in going to graduate school in pure mathematics and wants to learn about a potential research area. In the second part of this summary I take the opportunity to share the answers that I have developed to three of the questions that I am most commonly asked when a stranger or non-academic acquaintance learns of my profession.

Part 1: To the Curious Undergraduate

Either late in your undergraduate career or early in your graduate career you will take a course in topology in which you will learn about qualitative properties of mathematical objects called topological spaces. You already know some topological spaces such as the real number line, a sphere, a torus, or a mobius band. In graduate school you will be exposed to more topological spaces such as a klein bottle, the real projective space, manifolds, G-spaces, and spectra. It turns out that learning about advanced topological spaces is really hard. It becomes more difficult than abstract algebra. So, algebraic topologists have decided to use abstract algebra to learn about topological spaces. We realized that we could cleverly assign groups to spaces and gain a fair amount of information about spaces by studying these assigned groups instead of the spaces themselves. We call such groups 'invariants' of the spaces.

In graduate school I studied one type of invariant with the idea that if we can attain a better understanding of the algebra side of things then we can learn more about the topology side of things. But I did not study the groups that you learned about in abstract algebra. As topological spaces become more complicated plain old groups no longer suffice. We cannot gain enough information about the fancy topological spaces by assigning groups to them. We need invariants that encompass more information.

In the niche of equivariant stable homotopy theory one goal is to learn about topological spaces called G-spectra and their more elaborate siblings G-ring-spectra. One invariant for G-spectra is called a Mackey functor. The analogous invariant for G-ring-spectra is a Tambara functor. A Mackey functor is an incredibly robust group. If a group is a single gate at an airport, then a Mackey functor is an entire terminal. A Tambara functor is a Mackey functor with extra stuff added in, just as a ring is a group with extra stuff (notably the additional operation, multiplication).

I spent the bulk of my graduate career studying Mackey functors and Tambara functors. In truth I spent most of my time just trying to understand them by building examples and reading articles about them. But my end goal was to develop a new packaging for Mackey functors and Tambara functors that creates a new relationship between them. A better relationship between these algebraic objects will give us a better understanding of the relationship between the complicated topological things, G-spectra and G-ring spectra.

Part 2: Commonly Asked Questions

I will next address three of the most popular queries that arise when I discuss mathematics and my research with someone without an advanced degree in the subject.

Question 4.1. What are the "real-life" applications for Mackey functors and Tambara functors?

In truth, I do not yet know of any "real-life" applications of Mackey functors and Tambara functors, but I am also not currently interested in finding any. I like pure mathematics because I do not have to apply it to real-life. I like that I can escape to an abstract world and discover new mathematics without having to justify its existence. I can focus solely on the mathematics. However, though no real-life applications to my project currently exist, this may not always be the case. Mathematicians often develop theory and find an application for it later. For example, a major concept in algebraic topology is homology. Homology was developed in the late 1800s as a way to analyze mathematical shapes and spaces. But in the last decade we have adapted homology into a tool for analyzing big data. Perhaps we will discover some applications of Mackey functors and Tambara functors in the future, or perhaps studying them will lead to the development of other applicable mathematical concepts. Although my interest in Mackey and Tambara functors remains purely mathematical, I think it would be awesome if they helped create some new technology or played a role in the cure for a disease at some point in the future.

Question 4.2. Why do you like researching such an abstract topic?

I like math and specifically algebraic topology because I enjoy the mental challenge it presents to me. I love learning and growing by puzzling through difficult and abstract concepts. Working on this project felt like an intense training session for my brain. I dove into a dense world of functors, maps, actions and spectra. For months I continuously tried and failed to comprehend Mackey functors, Tambara functors, and how they best fit together. It was infuriating, but eventually I figured it out. Then I studied it some more until I could contribute to the field. That feeling of clarity and of finally understanding these objects was amazing. It made the struggle and frustration worth it!

Question 4.3. What good will come out of your research?

First, my research helps move the field of algebraic topology forward. It gives a new perspective on Mackey and Tambara functors, which is needed because Mackey and Tambara functors are particularly abstract mathematical objects. This new perspective will help topologists better understand these object which in turn will help us use Mackey and Tambara functors to create new mathematics.

But one of the more surprising outcomes of this project arose from the confidence and growth mindset that I developed from overcoming its challenges. Looking back on the time I spent studying Mackey and Tambara functors, I realize the value in all of the failed ideas and wrong turns that I took along the way. Of course these felt exasperating at the time, but they proved crucial to the research process. By learning what did not work and, more importantly, why it did not work, I gained a deeper understanding of the project. Moreover, knowing that I can understand and build abstract objects such as Mackey functors and Tambara functors granted me confidence in my ability to solve other difficult problems. Whenever I am stuck or frustrated I remind myself, "If you can understand Mackey and Tambara functors, then surely you can figure this out!" I have even applied this mentality to other research projects and academic challenges. For example, when I started working at Elon I had to teach a basic introductory statistics course that I never took in college. I needed to learn how to teach statistical concepts such as confidence intervals, hypothesis tests, and the Central Limit Theorem. The confidence and growth mindset that I gained through this research project were instrumental to my ability to overcome this challenge and successfully teaching the course.

In conclusion, the abstract and technical aspects of this project make it frustratingly difficult to convey its beauty to a general audience. But I hope that I have demonstrated why I find this research so provocative. It is a fun challenge that improves my ability to think critically and learn. Moreover, despite the lack of current real-life applications of Mackey functors and Tambara functors, the scientific community still benefits from studying them. Understanding G-spectra through Mackey and Tambara functors pushes the field forward and may one day provide a path towards future studies and applications. Indeed the contributions of this project both to my personal growth and to science and mathematics have proven to be subtle yet influential.

References

[Swi14] Taylor Swift, *"Blank Space", 1989*, Big Machine Records, 2014.

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