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# A user's guide: Dynamics and fluctuations of cellular cycles on CW complexes

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Contents



## 1. Key insights and central organizing principles

In [**CCK**], we consider the stochastic motion of subcomplexes in a CW complex, and explore the implications on the underlying space. This is a direct generalization of a random walk on a graph, and allows a higher-dimensional notion of electrical current, known as extended empirical or stochastic current, to be defined. Empirical currents provide a unique link between the random process and the topology of the complex. The ideas underlying empirical currents are a marriage of Langevin dynamics on smooth manifolds and discrete dynamics of random walks on graphs. In the manifold setting, the random process we consider is easier to express and the ideas behind empirical currents are more intuitive. This is in contrast to the discrete setting, where the precise formula for average current is simpler to state. In this user guide, we will focus on both cases and play them off one another to gain better intuition for the random processes under study, as well as their topological implications.

The first rigorous topological study of empirical current was for a particle taking a random walk on a graph in [**CKS13**]. For a particle undergoing stochastic motion on an arbitrary smooth manifold, current was first considered in [**CCMT09**]. However, both of these studies were for the motion of 0-dimensional objects. The focus of [**CCK**] is to analyze the motion of cycles of arbitrary dimension on a CW complex.

**1.1. Random walks on graphs.** Let $X$ be a connected graph, by which we mean a connected, one-dimensional CW complex. Consider a particle taking a random walk on $X$. The particle sits on the 0-cells, or vertices of $X$, and after waiting some random time, hops across a 1-cell, or edge, to an adjacent vertex. This defines a Markov process with state space given by the set of vertices, and transitions between states occur through the edges. The rate at which the particle moves is determined by external data. The typical set-up is to equip each vertex $i$ and each edge $\alpha$ with real numbers $E_i$ and $W_\alpha$. If $\alpha$ is an edge connecting vertices $i$ and $j$, then the rate at which the particle will hop from $i$ to $j$ across $\alpha$ is given by

$$H_{ji}(\beta) = e^{-\beta(W_\alpha - E_i)}\,, \tag{1}$$

where $\beta$ is a fixed positive real number[1]. We assemble these rates into a matrix $H = [H_{ij}]$ acting on $C_0(X;\mathbb{R}) = Z_0(X;\mathbb{R})$. The probability distribution of a particle taking a random walk on $X$ starting at vertex $i$ is governed by the Kolmogorov, or Fokker-Planck, equation

$$\frac{dp}{dt} = Hp \qquad p(0) = i\,, \tag{2}$$

for $p = p(t) \in C_0(X;\mathbb{R})$. When $p(t)$ is normalized, so that $\sum_{i\in X_0} p_i(t) = 1$, Eq. (2) governs the flow of probability on the graph.

It should be stressed that the homology class doesn't change throughout the random walk. In fact, one can (and should) think of 'hopping' from $i$ to $j$ across $\alpha$ as adding $\partial\alpha$ to $i$: $j = i + \partial\alpha$. This idea of hopping being the same as adding boundaries of cells one dimension higher is crucial for the more general situation to follow. The time which we let the particle evolve is known as the *evolution time* and is denoted by $\tau$. We are interested in an explicit formula for the particle's trajectory under various limits on the Markov process.

KEY IDEA 1.1. *The* average current *is given by the real homology class of the particle's trajectory on $X$, divided by the evolution time.*

It is easiest to see why this homology class deserves to be called current in the case of a manifold, which we now discuss.

**1.2. Langevin dynamics.** The Kolmogorov equation given above governs distributions of trajectories rather than describing a particle's actual trajectory. Let us spell out this stochastic motion in detail for manifolds.

[1] It can be shown that the rates for such a process can always be written in this form.

Fix a smooth, compact, Riemannian manifold $(M, g)$, together with a Morse function $f : M \to \mathbb{R}$. We also equip $M$ with a *stochastic vector field* $\xi$. By this, we mean a time-dependent vector field of $M$ satisfying Gaussian, Markovian statistics. This means that $\xi(t)$ is not only a vector field on $M$, but is also a Gaussian random variable in the sense of probability theory. Its first two moments are

$$\langle \xi(x,t) \rangle = 0\,, \quad \langle \xi(x,t)\xi(x,t') \rangle = \beta^{-1}\delta(t-t')g(x)\,, \tag{3}$$

where $g$ is the Riemannian metric. The delta-correlation in time reflects the Markovian property, and since the variable is Gaussian, the first two moments determine all the higher moments. A particle on $M$ will then move according to the Langevin equation

$$\frac{dx}{dt} = u(x,t) + \xi(x,t), \tag{4}$$

where locally, $u(x,t) = -\nabla f(x,t)$ is given by the gradient flow. The main point is that the stochastic motion is controlled by a deterministic piece and a stochastic piece.

A solution to Eq. (4) is a stochastic trajectory, and is represented by a path $\eta : [0, \tau] \to M$. For long times $\tau$, one can assume the path is closed [**CCMT09**, p. 6], and so the trajectory can be represented by $\eta : S^1 \to M$. This gives rise to a class in the real-bordism homology of $M$

$$Q_{\tau,\beta}(u) = \tfrac{1}{\tau}[\eta] \in H_1(M;\mathbb{R})\,,$$

known as the *average empirical current density associated to* $\eta$ *and duration* $\tau$.

The prototypical example from which this terminology is motivated is that of an electron in an electrical wire $M = S^1 \times D^2$, as in Figure 1. Connecting the wire to a battery gives rise to the deterministic term $u$, and random collisions with other particles and impurities in the wire give rise to the stochastic vector field $\xi$. For an oriented cross-section $\alpha : \{p\} \times D^2 \to S^1 \times D^2$, the *current at* $\alpha$ is the number of charged particle crossings at $\alpha$ per unit time. This is the (signed) number of crossings of the electron through $\alpha$, and is given by the intersection pairing $[\eta] \cdot [\alpha] = N_+ - N_-$, where $N_\pm$ is the number of crossings with positive/negative orientation. In this same sense,

$$\tfrac{1}{\tau}[\eta] \cdot [\alpha] \in \mathbb{R}$$

is the *average current at* $\alpha$ associated to the electron. The actual average current at $\alpha$ is a sum over all electrons.

If the manifold is one-dimensional, then the Morse function $f$ determines a graph structure on $M$[2], as shown in Figure 2. The vertices are given by the local minima $\{i, j, k\}$ and the edges by the local maxima $\{\alpha, \gamma\}$. Initially, a particle on $M$ will fall into a local minimum under the gradient flow; suppose it is $i$. If the

[2]In general, a Morse function gives rise to a CW decomposition as follows. The open cells are indexed by the local extrema of $f$, and are explicitly given by the unstable manifolds: the points which flow out of an extremum under the (negative) gradient flow of $f$.

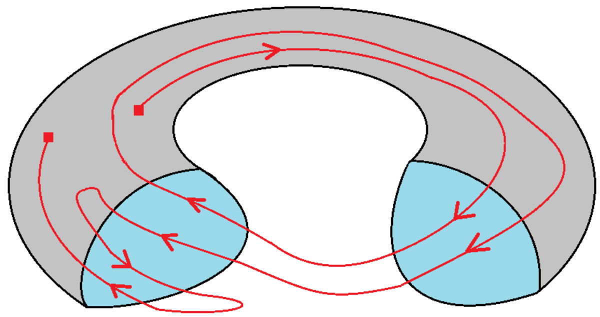

FIGURE 1. Traditional electric currents as considered in [**CCMT09**]. The stochastic trajectory of an electron is shown in red, and two cross-sections are shown in blue. The number of intersections is different, but the signed intersection indices agree.

vector field $\xi$ becomes large enough (which is rare since it is mean zero), it can push the particle out of the minimum and up towards a maximum. Suppose that $\xi$ is so large that the particle crosses the maximum at $\alpha$. It will then fall down into the minimum at $j$. This process of $\xi$ moving the particle from one minimum to another is identical to the particle jumping across an edge on a graph (and is precisely what is happening on the associated CW complex). The rate at which a hop occurs is governed by Eq. (1), with $E_i := f(i)$ and $W_\alpha := f(\alpha)$. In general, the larger the difference $f(\alpha) - f(i)$, i.e. the 'higher' the barrier, the less likely the particle is to cross $\alpha$. It is interesting to note that the rate at which a jump from $i$ to $j$ occurs does not depend on $j$, as seen in the rate of Eq. (1), as well as the intuitive picture of Figure 2. Furthermore, this picture makes it clear that the absolute barrier height $f(\alpha)$ does not determine the rates, but rather the relative differences $f(\alpha) - f(i)$.

**1.3. Extended empirical currents.** The additional complexity of stochastic motion in higher dimensions is not due to the ambient space in which the random motion occurs. Rather, it is because of the moving objects themselves. Unlike [**CKS12**] and [**CKS13**] which studies the motion of 0-dimensional objects, [**CCK**] studies the stochastic motion of objects of arbitrary dimension (hence the word 'extended'). With the intuition of particles moving on graphs and manifolds, we now describe the main stochastic process on CW complexes studied in [**CCK**].

Fix a finite, connected CW complex $X$ of dimension $d$, and an initial cycle $\hat{x} \in Z_{d-1}(X;\mathbb{Z})$. This cycle should be thought of as an initial condition, analogous to the starting vertex of a random walk on a graph. The initial cycle can 'hop'

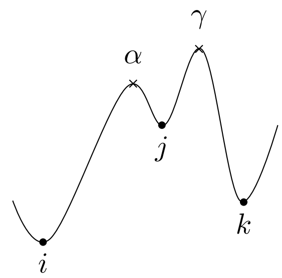


FIGURE 2. The height function on a one-dimensional manifold. The local minima, marked with dots, give rise to 0-cells and the local maxima, marked with crosses, yield the 1-cells.

across a $d$-cell to form a different $(d-1)$-cycle, given by adding the boundary of that $d$-cell. This is just as how a particle hops across an edge on a graph. By prescribing real numbers $E_i$ and $W_\alpha$ to each $(d-1)$ cell $i$ and $d$-cell $\alpha$, we can form rates which govern how the cycle will move on $X$, analogous to Eq. (1). Rigorously, the state space of this Markov process in higher dimensions consists of the set of real $(d-1)$-cycles homologous to $\hat{x}$. An elementary transition from state $z$ to $z'$ occurs if the cycles differ by adding or subtracting a single boundary of a $d$-cell. Generic transitions are compositions of elementary ones. The *average empirical current associated to $\hat{x}$ is*

$$Q_{\tau,\beta}(E,W) = \tfrac{1}{\tau}[x_\tau] \in H_d(X;\mathbb{R})$$

where $[x_\tau]$ is the homology class of the cycle after evolving for time $\tau$. We are interested in explicit formulas for this homology class.

The intuition behind the process and formula for current described above arise from the smooth picture. To play the two situations off each other, suppose the CW complex arose from a Morse decomposition on a smooth, Riemannian manifold $M$. Take the initial cycle to be a closed submanifold $\eta_0 : N \to M$. Initially, $\eta_0$ will evolve deterministically according to $-\nabla f$, and will tend to the $(d-1)$-skeleton of $M$ as determined by $f$ (this is the definition of the CW structure). The stochastic vector field $\xi$ will allow the cycle to fluctuate in some small neighborhood of the $(d-1)$-skeleton of $M$ ('small' since $\xi$ is mean-zero). On longer time scales, $\xi$ can push a segment of the evolved cycle $\eta_t$ and move it off the $(d-1)$-skeleton, into a $d$-cell. If the random field $\xi$ is large enough, this segment of $\eta_t$ can move up to a critical point of index $d$, against the gradient flow, and cross the critical point. This is completely analogous to the particle on a 1-dimensional manifold getting pushed up towards, and eventually crossing the local maxima. For either a particle or extended object, once it crosses the critical point, the gradient flow will push it back down into the $(d-1)$-skeleton; nothing special is needed of the noise for the object to 'relax', or literally flow, back into the $(d-1)$-skeleton. Throughout this process, the homology class never changes, since we have only added the boundaries of cells to it during its evolution. The *average empirical current associated to* $\eta_0$ is

$$Q_\tau(f) = \tfrac{1}{\tau}[\eta_\tau] \in H_d(M;\mathbb{R}),$$

where $\tau$ is the duration for which we let the cycle evolve and we use real bordism homology.

**1.4. A formula for average current.** The stochastic dynamics described above, both smooth and discrete, as well as the average current, are only interesting if the Morse function $f$, or the numbers $\{E_i, W_\alpha\}$ vary in time. Indeed, if the rates are fixed, an object will tend to the configuration of minimal energy (value of $E$ or $f$) and with overwhelming probability, remain there, resulting in zero current.

KEY IDEA 1.2. *Current generation can only occur if the parameters vary in time.*

The situation is more interesting if we drive the system by varying the numbers $E$ and $W$ periodically in time. The space of parameters $\mathcal{M}_X$ parameterizes the possible rates for the Markov process. We are interested in the limit of observing many periods of $\gamma$, and implicitly take $\tau = N\tau_D$, where $N$ is a large integer and $\tau_D$ is the period of $\gamma$.

Practical formulas for the average current can be obtained after employing two limits. The first is the low-temperature limit, in which $\beta \to \infty$. This is also known as the low-noise limit, since the dynamics become more and more deterministic as $\beta$ grows (see Eq. (3)). The other limit is the *adiabatic* limit, in which $\tau_D \to \infty$. The term adiabatic appreciates the fact that the parameters change extremely slowly in time.

For $\tau_D$ sufficiently large, the average current can be written explicitly in terms of 2 ingredients. The first is Kirchhoff's higher dimensional network theorem of [**CCK15a**], which gives the unique solution to Kirchhoff's laws in an electrical network. The second appears in [**CCK15b**], and is a higher dimensional notion of the Boltzmann distribution. We postpone their discussion to the next section. For now, we state the main result of [**CCK**]:

THEOREM 1.3 ([**CCK**]). *Let $X$ be a finite, connected CW complex of dimension $d$. For a sufficiently generic $\gamma$,*

$$\lim_{\tau_D\to\infty} Q_{\tau_D,\beta}(\gamma) = \int_0^1 K(\dot\rho^B)dt\,,$$

*where $K$ is the Kirchhoff operator and $\rho^B$ is the Boltzmann distribution. Furthermore, in the low-temperature, adiabatic limit, the average current quantizes:*

$$\lim_{\beta\to\infty}\lim_{\tau_D\to\infty} Q_{\tau_D,\beta}(\gamma) \in H_1(X;\mathbb{Z}[\tfrac{1}{D}]) \subset H_1(X;\mathbb{R})$$

*where $D$ is determined by combinatorial data of $X$.*

## 2. Metaphors and imagery

In this section, we visualize the concepts described in the previous section. We first give imagery concerning the stochastic process on general CW complexes, together with their smooth generalizations. We then discuss the two key components to the average current: Kirchhoff's network theorem/solution and the Boltzmann distribution. We discuss the inherent geometric nature of these two pieces and show how they can be used to think about the average current.

**2.1. Visualizing the stochastic process.** A simple picture of a transition in higher dimensions is displayed in Figure 3 for a 2-dimensional CW complex, although the general picture is very similar. Recall that to interpolate between the smooth and discrete cases, we describe a CW complex arising from a Morse decomposition. An elementary transition, as described in the previous topic, is shown in Figures 3a-3d. The cycle $\hat{x}_0 = i + j$ shown evolves in the manifold by jumping 'off' $i$ and 'across' $\alpha$ to

$$\hat{x}_1 = \hat{x}_0 + \partial\alpha = \hat{x}_0 + j - i = 2j\,.$$

This is one aspect of working in higher dimensions which is significantly different from the graph case. An elementary transition on a graph only requires an edge, or 1-cell. On a CW complex of arbitrary dimension $d$, we must specify a $d$-cell to hop across and a $(d-1)$-cell to hop 'off'. It is important to note that only the first and last figures take place on the CW complex, whereas the intermediate transition lies in the smooth manifold. We only use the smooth picture for motivation, so we think of this transition occurring instantaneously on the CW complex.

If $\alpha$ had more boundary components, the situation would be more complex. For example, take $\partial\alpha = m_1 + \cdots + m_k$ with $\hat{x}_0 = \sum_j n_j m_j$ for some $(d-1)$-cells $m_j$ and integers $n_j$. In the above scenario, in which $\hat{x}_0$ moves off $m_i$ along $\alpha$, the evolved cycle would be

$$\hat{x}_1 = \hat{x}_0 - n_i\partial\alpha = \sum_{j\neq i}(n_j - n_i)m_j\,,$$

so that $\hat{x}_1$ has no incidence with $m_i$, and the incidence $x_0$ has with $i$ has been subtracted from the rest of the incidences.

We now turn our attention to the visualization of the current and the pieces it is constructed from: the Kirchhoff solution and the Boltzmann distribution.

**2.2. Spanning trees and co-trees.** Kirchhoff constructed a solution to the network problem on graphs (see [**CKS13**] and [**CCK15a**]) using spanning trees. A spanning tree in higher dimensions can be thought of as an appropriate truncation or approximation to the CW complex $X$. The point is that spanning trees do not have any $d$-cycles, but still contain enough of $X$ to have all the rational homology of $X$ in lower degrees. From the viewpoint of current, they

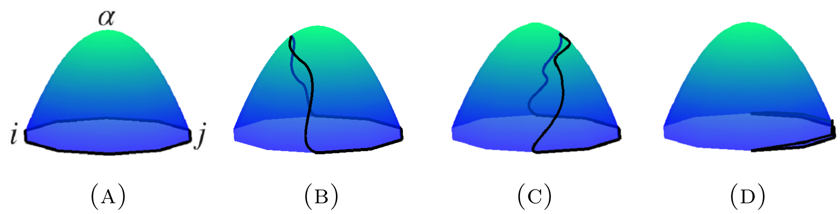


FIGURE 3. An elementary transition on a CW complex. A 2-cell $\alpha$ is shown, given by the local maximum; the left side of its boundary is a 1-cell $i$ and the right is the 1-cell $j$. The transition occurs from the $i$ to $j$ along $\alpha$.

satisfy one crucial property: let $b \in B_{d-1}(X;\mathbb{Q})$ be any boundary of $X$ and let $T$ be any spanning tree. There is a unique $d$-chain $K_b^T \in C_d(T;\mathbb{Q})$ such that[3]

$$-\partial_T K_b^T = -\partial K_b^T = b.$$

This generalizes the fact that on a graph, every spanning tree contains a unique path between any two vertices (their difference being a boundary). The solution to the network problem is given by taking a weighted sum of such operators

$$K = \tfrac{1}{\Delta}\sum_T w_T K^T : B_{d-1}(X;\mathbb{Q}) \to C_d(X;\mathbb{Q}).$$

The other ingredient is the Boltzmann distribution. Originally defined as an energy distribution for particles in a gas, it was shown in [**CCK15b**] that the Boltzmann distribution can also be used to describe harmonic forms on CW complexes. The distribution is written as a sum over certain subcomplexes known as spanning co-trees. These subcomplexes are again approximations to $X$, and contain enough of $X$ to reproduce its rational homology in degree $(d-1)$. They are so useful because they satisfy the following. Let $[x] \in H_{d-1}(X;\mathbb{Q})$ be any homology class of $X$ and let $L$ be any spanning co-tree. Then there is a unique cycle $\psi_L([x]) \in Z_{d-1}(L;\mathbb{Q})$ representing $[x]$. Very roughly, a spanning co-tree should be thought of as a generalization of a vertex on a graph to an arbitrary CW complex. On a connected graph, the spanning co-trees are precisely the vertices, and $\psi_L([x])$ is the vertex itself.

**2.3. Visualizing current generation.** To describe the average current and the governing quantization results, we discuss[4] a simple example on $S^1$. We take the CW structure to have two 0-cells (vertices $i$ and $j$), and two 1-cells (edges $\alpha$ and $\delta$), as in Figure 4. Furthermore, assume we have taken both the adiabatic and low-temperature limits to simplify the discussion. We use a periodic driving protocol of good parameters, so that if at any time the vertex energies agree, the edge energies must be distinct, and vice-versa. In particular, take a

[3]The minus sign on $\partial$ is physically motivated, and does not affect the relevant algebra. The topologist can omit it.

[4]This is a restatement of a discussion in [**CKS12**].

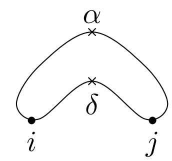


FIGURE 4. The height function on $S^1$, giving rise to a CW decomposition, with parameters shown as in segment (ii) of the driving protocol.

periodic driving protocol $\gamma$, starting with $E_i < E_j$ and $W_\alpha < W_\delta$, split into 4 segments:

(i) vary $E$ so that $E_i > E_j$, keeping $W$ fixed,
(ii) vary $W$ so that $W_\alpha > W_\delta$, keeping $E$ fixed,
(iii) vary $E$ so that $E_i < E_j$, keeping $W$ fixed, and
(iv) vary $W$ so that $W_\alpha < W_\delta$, keeping $E$ fixed, and returning to the original parameters.

Initially, the particle will fall to vertex $i$, since it has the lowest energy. As $E_i$ approaches $E_j$ on segment (i), the particle will hop back and forth between $i$ and $j$ across the edge with lowest energy $\alpha$. Once condition (i) is satisfied, the particle will sit at node $j$, and hence will have travelled a net distance of one across $\alpha$ and zero across $\delta$. On segment (ii), since the vertex energies remain fixed, the particle will not move. Once $E_i$ and $E_j$ vary on segment (iii), the particle will hop back and forth between the vertices, this time crossing $\delta$ a net total of once, while not crossing $\alpha$, and finishing at vertex $i$. No motion will occur on segment (iv), because the vertex energies are unchanged. Therefore, over one driving protocol, the particle will perform one full rotation around $S^1$. The homology class of this trajectory $Q(\gamma)$ clearly represents the generator of $H_1(S^1;\mathbb{Z}) \subset H_1(S^1;\mathbb{R})$ and the average current is 1. Since the average current lies in the integer lattice of the real homology group, we say the current is *quantized*. This is the idea which stands behind the quantization results of current. Moreover, this phenomenon is generic for graphs, as shown in [**CKS13**].

The key idea which underlies this motion (and many phenomena in physics) is that, as the energies vary, the particle is always tending to or 'following' the vertex with lowest energy through the spanning tree of minimal energy, as seen in the motion on the graph of Figure 4. On segments (ii) and (iv), $E$ has a unique minimum, and hence there is a preferred vertex or spanning co-tree in $X$. On segments (i) and (iii), $W$ has a unique minimum, so we can construct a preferred spanning tree. In our example, this tree consists of a single edge, and contains both vertices. On a more complicated graph, the particle would traverse the unique path in $T$ given by $K^T_{j-i}$, connecting vertices of minimal energy $i$ and $j$.

The situation in higher dimensions is analogous. The cycle will jump from spanning co-tree to spanning co-tree by traversing spanning trees. The evolving cycle is always attempting to minimize $\sum_{b \in L} E_b$, thought of as the 'energy' of the cycle. As in the graph case, one can always form a decomposition of the periodic driving protocol $\gamma$ to alternate between segments of type $U$, with a unique co-tree, and type $V$, with a unique tree. For example, consider a driving protocol $\gamma$ going from type $U$ to $V$ to $U$, with associated spanning co-tree $L$, spanning tree $T$, and spanning co-tree $L'$, respectively. On the $U_L$ segment, the initial cycle tends to the cycle $\psi_L([\hat{x}])$ supported on the spanning co-tree $L$. The cycle remains in this configuration with overwhelming probability until the parameters change further. This occurs on the $V_T$ segment, where the cycle will transition from $\psi_L([\hat{x}])$ to $\psi_{L'}([\hat{x}])$ within the spanning tree $T$. In fact, the transition occurs through the unique $d$-chain $K^T_{L-L'}$, and ends once the cycle becomes $\psi_{L'}([\hat{x}])$. These types of transitions keep happening until a full period of the driving protocol occurs.

The average current generated by this process can be written as a sum over segments alternating between type $U$ and type $V$. On type $U$ segments, the object will remain on the unique spanning co-tree and no current will be generated. On type $V$ segments, current is generated by the motion along spanning trees. The coefficients which appear in the formula for the trajectory, or average current, $Q(\gamma)$ depend entirely on these subcomplexes to which the motion is restricted. The main result of [**CCK**] is that, in the long time limit, $Q(\gamma)$ will have rational incidence with each of the $d$-cells in $X$. That is, it will form a rational $d$-dimensional homology class, as opposed to a real homology class. This is in contrast to the main result on graphs [**CKS13**], in which the current is integer-valued, as in the previous example on $S^1$. The reason for this difference is due to a variety of factors, notably the order of torsion subgroups which are non-trivial in higher dimensions. This is further complicated by the more elaborate structures, like a vertex compared to a generic spanning co-tree.

## 3. Story of the development

The ideas studied in [**CCK**] are a culmination of my work as a PhD student at Wayne State University with my two advisors and co-authors, Vladimir Chernyak and John Klein. I began the project as soon as I entered the PhD program in January 2012 and finished with my graduation in March 2016. The work was split into three main pieces: the higher Kirchhoff theorem [**CCK15a**], the higher Boltzmann distribution [**CCK15b**], and coalescing them into a result on stochastic dynamics [**CCK**].

The background work began the year before I started working with John and Vladimir. In the summer of 2011, they visited Los Alamos National Laboratory, and together with Nik Sinitsyn, wrote both statistical mechanics and topology papers, in which they defined stochastic current for random walks on graphs. This work was interesting for a variety of reasons, but at least superficially, it mixed algebraic topology with statistical mechanics in a completely novel way. Their main result was to relate stochastic dynamics to two classical results in physics, the Kirchhoff network problem and Boltzmann distribution, to obtain an *integral* homology class, instead of a real homology class as was expected. Interestingly, the integral homology class had been observed in a variety of experiments on molecular motors, ratchets, and in other settings well before they formulated and proved their result. The fact that this result had experimental evidence before it was made precise is something which can't be said for many theorems in topology. Furthermore, their papers included various conjectures about points moving in higher-dimensional CW complexes, but not about the motion of higher-dimensional subcomplexes. This formed the starting point of my contributions.

When I started in January 2012, we began by attempting to generalize Kirchhoff's network theorem to CW complexes. This required the notion of spanning tree for a CW complex of arbitrary dimension. While similar objects had been defined by combinatorialists prior to our work, we were unable to find a definition suitable for our context. So, we defined a higher spanning tree based on what we needed to solve the higher Kirchhoff problem. In March 2012, we discovered a purely algebro-topological proof of the classical theorem had been given in 1961 [**NS61**]. This was of tremendous help, since we could follow their ideas and proofs of the classical Kirchhoff theorem in an attempt to prove the higher Kirchhoff theorem, with the exception of a few key lemmas. It turned out that the one-dimensional formulas of [**NS61**] generalized correctly only when the CW complex $X$ had no torsion in its homology, e.g., a graph. We discovered through trial and error that by appropriately introducing the torsion factors $\theta_T$ of Section 2 and modifying the standard inner products on $B_*(X)$ and $C_*(X)$, we obtained a formula that worked in general, and importantly, reduced to the classical result. In May 2012, I travelled to LANL as a summer student, working on a different but related project. There we finished the main proofs and the final form of [**CCK15a**] took shape.

From the first week of speaking with him, Vladimir had been asking me to work out a higher dimensional Boltzmann distribution. It was very unclear at first what properties such an operator should satisfy. We tried to define it using peculiar orthogonality conditions on $C_*(X)$ and $B_*(X)$, with various inner products similar to the Kirchhoff problem. We attempted to solve the problem using spanning trees, but this quickly failed. We next tried to solve the problem algorithmically, writing equations for every codimension one cell of the complex and then solving these equations in the low-temperature limit. This led to trivial (and incorrect) solutions for spaces with torsion in their homology such as Moore spaces. I distinctly recall Vladimir expressing his concern that maybe such a closed form could only be written in the low-temperature limit, and did not exist in general. Given how amazing his intuition about the project (and in general) was up to that point, this was worrisome.

Finally, in January 2014 I was able to write down a simple statement of the Boltzmann problem. The solution to the problem took shape in March 2014 by thinking back to the classical Boltzmann distribution, which is written as a sum over vertices in a graph. Based on our definition of a spanning tree, we asked how one classifies a vertex of a graph homologically. This consideration led us to define a spanning co-tree for a CW complex, so named because they are dual to spanning trees[5]. Our homological approach may seem strange given the enormous discrepancy between graphs and generic CW complexes (e.g., the complexity of general attaching maps as compared to those of graphs), but it provided the solution. We knew they would need to be appropriately weighted and summed for the desired formula to work out correctly.

When we wrote down the Boltzmann distribution, we were unaware of how strong the duality was between it and the Kirchhoff problem. This came to light in April 2014 while we were attempting to prove that our proposed distribution was in fact correct. After several manipulations, we transformed the problem into a statement of linear algebra, for which we were sure the answer was known. We eventually came across the theory of pseudo-inverses. Pseudo-inverses provide one-sided inverses for maps which fail to be either injective or surjective (or both). It turns out that pseudo-inverses have an explicit expression as a sum, and this sum aligned perfectly with our proposed sum over spanning co-trees. Even more, by applying the pseudo-inverse construction to the boundary map $\partial$ on $C_*(X)$, we reproduced our prior solution to the Kirchhoff problem. It was both amazing and extremely satisfying that the two main components needed for average current were really different manifestations of the same thing.

By the fall of 2014, the individual pieces for what we thought would give rise to the formula for current were in place. All that remained was to rigorously define the stochastic process that our current described. This turned out to require substantial effort; even rigorously defining what the state space should be took several months worth of discussions and tweaking. In the earlier work of [**CKS13**], the graph on which the dynamics were evolving was the natural and correct choice. The higher dimensional case was dramatically different, in that the CW complex was not adequate. We discovered this fact quickly, since the Fokker-Planck operator (see Eqn. 2 in Topic 1) on a CW complex does not conserve probability. That is, it does not govern the evolution of a probability distribution. By 'enlarging' the state space to an infinite graph, the operator worked just as it should.

Once all the pieces were in place, it did not take very long to prove the main quantization result. This is often how these things work: put a lot of effort into the supporting ideas and the main result might just 'fall out'. The paper [**CCK**] was written in the first few months of 2016 in conjunction with my thesis. All three of us were happy with how the pieces of the project fit together so nicely to describe a new approach to stochastic dynamics.

[5]It has been only recently pointed out that our spanning co-trees have been defined in other contexts.

## 4. Colloquial summary

Imagine standing on a busy street with cars driving by. If you own a business on the street, work road construction, are a city planner, or work in the government traffic office, then the rate and actual number of cars driving by could be of great interest to you. The rate at which cars drive by depends on a variety of different factors, all of which can be described in the language developed so far in this user guide. In this section, we explore this example in detail in an attempt to connect the ideas of [**CCK**] with a real-world example.

The rate at which cars pass by is determined by a number of different features. Some of these features are completely determined ahead of time, that is they can be planned for, whereas others cannot. For example, the speed limit, the time of day, day of the week, and which businesses are nearby will all affect the rate at which cars pass. If we are interested in the rate as precise function of time, meaning on the level of seconds, then the traffic signals will have a tremendous impact.

There are also random effects for which we cannot prepare and are beyond our control. In this situation, random could mean someone oversleeping their alarm before work and having to speed, a broken traffic signal, or even more subtle, the different speeds at which different people drive. Furthermore, the actual road itself will factor into this rate as well. The more intersections a road has, the more traffic signals and possibly more pedestrian traffic there can be, decreasing the rate at which cars pass. On the other hand, a road with more lanes can increase this rate.

In terms of the language developed in the previous sections of this user guide, we have described deterministic and stochastic (or random) effects, both of which affect the rate at which cars pass by. The deterministic component consists of the steady pattern of the traffic light, which forces the traffic to move in a specific direction at certain times, and more generally, a person's desire to travel to a destination.

The stochastic component is mostly due to human drivers. Different people drive differently. Some go fast as soon as they see the green light, whereas others drive slower and with more caution. The varying driving styles, together with possible pedestrian traffic, force the flow of traffic to be slightly randomized, even though the traffic signals generally govern the flow of cars. Also the topology of the road, such as the number of lanes and number of intersections, has a great impact on the rate.

The main result of [**CCK**] is a statement about currents or rates under two important limits. The adiabatic limit is one in which the parameters vary slower and slower in time. In the example of traffic, this amounts to keeping the traffic lights greener for longer and longer. This would make the flow of traffic much more regular and consistent, since the cars no longer have to stop. They just keep driving at their normal pace.

The second limit is the low-noise limit, which takes the form of forcing the drivers to become more alike in their driving styles. Drivers will start to accelerate at the same pace, drive at the same speed, and drive less erratically. In a more modern situation, this could be achieved by placing more and more driverless cars on the road. This again serves to regulate the flow of cars. In these two situations, counting cars as they pass becomes much easier. Furthermore, it is much more believable that something meaningful about passing car rates can be said under these circumstances. The main quantization theorem of [**CCK**] metaphorically implies the number and rate of cars driving by will be the same for equal durations of time.

Higher dimensional currents can also be thought of in a similar example. In order to understand currents of extended objects, consider a team of runners moving along a marathon course. We are again measuring current by counting how many times the team runs by a particular spot. Each runner's desire to win serves as the deterministic component, driving them all. The way in which they run, at different paces, and in different ways around the course causes random behavior. The layout of the course in terms of turns and possible junctions plays the role of topology. The course can have forks and places where the path splits, but ultimately all paths lead to the finish line.

Suppose the team comes to a fork, and half run down one path and half down the other path. Each path was crossed by half the team, even though the team as a whole passed through the fork. Therefore, the 'current generated' by the team would be half on each leg of the fork, resulting in rational current. Note that any individual runner will only cross a single path of the fork, and therefore cross each leg either zero or one times, yielding integer current. While this is not precisely how rational quantization occurs in higher dimensions, it is the additional complexity of an extended object which gives rise to rational current for both our example and empirical currents.

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