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A user's guide: Landweber flat real pairs and $ER(n)$ cohomology

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Contents

This user's guide is about the paper *Landweber flat real pairs and $ER(n)$-cohomology* [**KLW16a**], joint with Nitu Kitchloo and W. Stephen Wilson.

1. Key insights and central organizing principles

1.1. Introduction. Algebraic topologists view the world (topological spaces) through the lens of algebraic invariants (cohomology theories). Two criteria along which cohomology theories are often judged are

(i) how much information they see and (ii) how computable they are.

These two criteria often pull in opposite directions. Given a generic cohomology theory, the first spaces one might try to compute its value on are some building blocks such as spheres S^k, projective spaces $\mathbb{RP}^k$ or $\mathbb{CP}^k$, Eilenberg MacLane spaces $K(G, q)$, or classifying spaces of various compact Lie groups BG. The goal of [**KLW16a**] is to compute the value of the cohomology theory Real Johnson-Wilson theory, $ER(n)$, on some of these spaces (about half of them). Before we get into the details, let's step back and build a case for why $ER(n)$ is a theory worth investigating, i.e. why it ranks highly in both criteria.

1.2. Detecting torsion. Computing the stable homotopy groups of spheres is a major motivating problem in algebraic topology. One way to get a handle on elements in π_*S^0 is to detect them using various cohomology theories. That is, given a multiplicative cohomology theory E, we may hope that some elements of π_*S^0 show up in the coefficients E_* via the image of the unit map $\pi_*S^0 \longrightarrow E_*$ We could then study how these spherical classes act on $E^*(X)$ for various spaces X.

Positive degree elements of the stable homotopy groups of spheres are all torsion. As such, for the above approach to yield any interesting information, it is necessary that some of these classes do not map to zero in E_*. This, for one, requires that E_* contain torsion and furthermore that the unit map not factor though any cohomology theories whose coefficients are torsion-free. This latter criterion rules out any *complex-oriented* cohomology theory, as for all such theories, the unit map factors through MU_*. Unfortunately, by far the most computationally accessible cohomology theories to date have been the complex-oriented ones[1]. So, how do we compute with non complex-oriented theories?

EXAMPLE 1. (KO **and** KU) Complex K-theory KU is complex-oriented, torsion-free, and very amenable to computations. It also carries an action of the group of order 2, C_2, which can be seen geometrically via the action of complex conjugation on vector bundles. Taking fixed points produces real K-theory, KO, which is no longer complex-oriented or torsion-free. In fact, KO detects the spherical class η in its Hurewicz image. What makes KO computable is that η generates *all* of the torsion and so KO may be computed from KU by building in η^r-torsion one r at a time using the η-Bockstein spectral sequence $E_1^{*,*} = KU^*(Z) \Rightarrow KO^*(Z)$.

This example supports the claim that KO is a cohomology theory which satisfies both conditions laid out at the beginning. Furthermore, it suggests we can find more such theories by looking for complex-oriented cohomology theories which carry group actions and taking their fixed points. Chromatic homotopy theory, a subfield of algebraic topology, provides many examples of complex-oriented cohomology theories, from which we pick a certain family. From here on, we work 2-locally. The examples in [**KLW16a**] are the Johnson-Wilson theories $E(n)$ [**JW73**] for $n \geq 1$. Their coefficients are torsion-free, given by

$$E(n)_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}], \quad |v_i| = 2(2^i - 1)$$

They all carry an action of C_2 which ultimately stems from complex conjugation, as with KU. In fact, $E(1)$ is $KU_{(2)}$.

[1] The complex orientation implies that E has a theory of Chern classes and in particular makes possible the immediate computation of spaces like $\mathbb{C}P^\infty$, $\mathbb{CP}^k$, $BU(q)$, and $B\mathbb{Z}/(p^k)$. There are many other computations that take more work. See e.g. [**RW80**, **RWY98**] for the complex-oriented cohomology of Eilenberg-MacLane spaces and much more.

Equivariant interlude. To distinguish a cohomology theory with C_2-action from an ordinary nonequivariant one, we will denote the former with boldface letters, such as $\mathbb{E}(n)$. As we will see later, there will be great value in letting our group act on the spheres. As such, given a (real) representation V, let S^V denote the one-point compactification as an equivariant space. In the case of C_2, all representations have the form $s + t\alpha$ where α denotes the trivial representation. We may now index our homotopy and cohomology groups not just over integers but over representations.[a] Define $\pi_{s+t\alpha}(\mathbb{E}(n)) := [S^{s+t\alpha}, \mathbb{E}(n)]^{C_2}$ and $\mathbb{E}^{s+t\alpha}(X) = [X, \Sigma^{s+t\alpha}\mathbb{E}]^{C_2}$.

[a] Example: the regular representation, $1+\alpha$ is exactly the complex plane with C_2 acting by complex conjugation, and $S^{1+\alpha}$ is the Riemann sphere.

As with KU, we may take the C_2-fixed points of $\mathbb{E}(n)$ to produce a family of interesting new cohomology theories.

DEFINITION 1. Define *Real Johnson-Wilson theory* to be the C_2-fixed points of Johnson-Wilson theory, $ER(n) = \mathbb{E}(n)^{C_2}$.

Just as with KU and KO, it turns out that $ER(n)$ is close enough to $E(n)$ to be computationally accessible but has just enough extra information (via torsion) to make it interesting.

THEOREM 1. *(Kitchloo-Wilson [**KW07a**]) There is a class $x \in \pi_\lambda ER(n)$ with $\lambda = 2^{2n+1} - 2^{n+2} + 1$ and a fibration $ER(n) \xrightarrow{x} ER(n) \longrightarrow E(n)$ which yields a Bockstein spectral sequence $E_1^{*,*} = E(n)^*(Z) \Rightarrow ER(n)^*(Z)$.*

KEY IDEA 1.1. *All of the torsion in the coefficients of $ER(n)$ is generated by a single class x. Thus, we may compute $ER(n)$-cohomology from $E(n)$-cohomology by building in x^r torsion for a single r at a time, as carried out by the Bockstein spectral sequence (BSS).*

1.3. Producing permanent cycles. We now have a potentially interesting cohomology theory, $ER(n)$, and a method to compute it from a more accessible cohomology theory, $E(n)$. The next step is to compute its value on some interesting spaces. We will describe the approach taken in [**KLW16a**], first via an example from [**KW08a**].[2].

EXAMPLE 2. ($ER(n)^*(\mathbb{R}P^\infty)$**, part I**) We have a spectral sequence which begins with $E_1^{*,*} = E(n)^*(\mathbb{R}P^\infty)$ and converges to $ER(n)^*(\mathbb{R}P^\infty)$. The $E(n)$-cohomology of $\mathbb{R}P^\infty$ is known[a] and so the next step is to compute

[2] The $ER(n)$ cohomology of projective spaces is not just a curiosity. In [**KW08a**], Kitchloo and Wilson used their computation of $ER(2)$-cohomology to demonstrate some nonimmersion results for real projective spaces—for example, that $\mathbb{RP}^{48}$ does not immerse in $\mathbb{R}^{84}$. The torsion in $ER(2)$ is key to their results which are undetectable by any complex-oriented theory.

Bockstein differentials. While there is an explicit formula for d_1, computing higher differentials is difficult. The computation becomes easier if we can identify some classes in $E(n)^*(\mathbb{R}P^\infty)$ which do not support differentials (i.e. permanent cycles). These are classes in $E_1^{*,*} = E(n)^*(\mathbb{R}P^\infty)$ which are in the image of the map $ER(n)^*(\mathbb{R}P^\infty) \longrightarrow E(n)^*(\mathbb{R}P^\infty)$.

A crucial observation is that both $\mathbb{R}P^\infty$ and $ER(n)$ are fixed points—that is, $\mathbb{R}P^\infty = \mathbb{C}P^{\infty C_2}$ and $ER(n) = \mathbb{E}(n)^{C_2}$. Hence, given an equivariant map $\mathbb{C}P^\infty \longrightarrow \mathbb{E}(n)$, we may take fixed points to get a map $\mathbb{R}P^\infty \longrightarrow ER(n)$. In other words, there is a homomorphism

$$[\mathbb{C}P^\infty, \mathbb{E}(n)]^{C_2} \longrightarrow [\mathbb{R}P^\infty, ER(n)] = ER(n)^*(\mathbb{R}P^\infty)$$

[a] $E(n)^*(\mathbb{R}P^\infty) = E(n)^*(B\mathbb{Z}/2) = E(n)^*[u]/([2]_F(u))$ where $[2]_F(u)$ is a certain power series computable using the formal group law for $E(n)$

This example can be summarized in the following key idea:

Key Idea 1.2. *To get ahold of classes in $ER(n)^*(Z)$, map Z into the fixed points of another space Y. Then the image of the homomorphism*

$$[Y, \mathbb{E}(n)]^{C_2} \longrightarrow [Y^{C_2}, ER(n)] \longrightarrow [Z, ER(n)] = ER(n)^*(Z)$$

provides classes in $ER(n)^*(Z)$ which are detected by permanent cycles in the Bockstein spectral sequence.

In the above example, we have $(Z, Y) = (\mathbb{R}P^\infty, \mathbb{C}P^\infty)$ and the map from $\mathbb{R}P^\infty$ to the fixed points of $\mathbb{C}P^\infty$ happens to be an equivalence, though in general this is not a requirement. We have now reduced the problem to computing the source of the composite homomorphism above, the equivariant $E(n)$-cohomology of $\mathbb{C}P^\infty$.[3]

1.4. Computing equivariant cohomology.

The computation of

$$[\mathbb{C}P^\infty, \mathbb{E}(n)]^{C_2} = \mathbb{E}(n)^{*+*\alpha}(\mathbb{C}P^\infty)$$

has two conceptual ingredients: *the projective property* and *the hat construction.*

1.4.1. ***The projective property.*** A key input that makes our C_2-equivariant computations possible is that our spaces Y above (such as $\mathbb{C}P^\infty$) have an important property with respect to $\mathbb{E}(n)$, first defined by Kitchloo and Wilson in [**KW07b**]. We will work through what this means in the case of $\mathbb{C}P^\infty$. We begin by noting that the forgetful map $\rho : \mathbb{E}(n)_{*(1+\alpha)} \longrightarrow E(n)_{2*}$ is an isomorphism [**HK01**]. From this, we may filter the C_2-CW complex $\mathbb{C}P^\infty$ skeletally and notice that the associated graded is given by $\mathbb{C}\mathbb{P}^k/\mathbb{C}\mathbb{P}^{k-1} = S^{k(1+\alpha)}$. It follows that the forgetful map ρ is an isomorphism on these pieces. Furthermore, the

[3] Recall that this is a bigraded object. We are especially interested in classes in integral degrees $* + 0\alpha$.

$\mathbb{E}(n)$-cohomology of these quotients forms the E_2-page of an equivariant Atiyah-Hirzebruch spectral sequence converging to $\mathbb{E}(n)^{*+*\alpha}(\mathbb{C}P^\infty)$. This spectral sequence turns out to collapse (for reasons similar to those in the nonequivariant case), and the isomorphism above may be boosted up to an isomorphism

$$\mathbb{E}(n)^{*(1+\alpha)}(\mathbb{C}P^\infty) \xrightarrow[\rho]{\cong} E(n)^{2*}(\mathbb{C}P^\infty)$$

So we can at least compute the 'diagonal', degrees $k(1+\alpha)$ part of $\mathbb{E}(n)^\star(\mathbb{C}P^\infty)$. As we will see from the hat construction, computing the diagonal degrees is enough to give us everything we need. Definition 1.4 of [**KLW16a**] defines what it means for spaces to be 'sufficiently projective' such that ρ is an isomorphism for them as well (proved in Theorem 1.5). Examples of spaces with this projective property include $\mathbb{C}P^\infty$, BU, BSU, $BU\langle 6\rangle$, and all Wilson spaces[4].

1.4.2. ***The hat construction.*** But we have not computed all of the bigraded ring $\mathbb{E}(n)^{*+*\alpha}(\mathbb{C}P^\infty)$, and what we're really interested in are classes in integer degrees since those are the ones which will map to $ER(n)^*(\mathbb{R}P^\infty)$. It turns out that for computing $\mathbb{E}(n)$-cohomology, the diagonal degrees are enough. As observed by Kitchloo and Wilson, the coefficients $\mathbb{E}(n)_\star$ contain a class y in degree $\lambda+\alpha$, where λ is the same integer as above, and *this class is invertible!* It follows that, given any class z in degree $s+t\alpha$, we may define the product $\widehat{z} := zy^{-t}$, which lives in an integral degree. This defines an injective ring (but not graded ring) endomorphism of $\mathbb{E}(n)^\star(-)$ given by 'hatting' each class, and allows us to move all diagonal classes to integral degrees.

Key Idea 1.3. *Given Y and Z as above, classes in $E(n)^*(Y)$ can transform into classes in $ER(n)^*(Z)$ as follows. Combining the projective property and the hat construction, we have*

$$E(n)^{2*}(Y) \xrightarrow{\rho^{-1}} \mathbb{E}(n)^{*(1+\alpha)}(Y) \xrightarrow{\widehat{\ }} \mathbb{E}(n)^{*(1-\lambda)}(Y)$$

We may then postcompose with the map induced by $Z \longrightarrow Y^{C_2} \longrightarrow Y$ to get a map

$$\psi : E(n)^{2*}(Y) \longrightarrow ER(n)^{*(1-\lambda)}(Z)$$

The source is computable and so this map produces classes in the target that are easily detected as permanent cycles on the E_1-page of the BSS.

1.5. Putting the pieces together. We return to finish off our example.

Example 3. ($ER(n)^*(\mathbb{R}P^\infty)$**, part II**) We have

$$E(n)^*(\mathbb{C}P^\infty) = E(n)^*[[u]].$$

[4]Wilson spaces are H-spaces whose p-local homotopy and homology are both free $\mathbb{Z}_{(2)}$-modules. In [**KLW16a**], they are the projective spaces Y whose fixed points we map into to compute the $ER(n)$-cohomology of some Eilenberg MacLane spaces. For more of their properties, see Wilson's thesis [**Wil73, Wil75**] as well as [**RW77**] (and [**BW07**] for a generalization)

We denote the image of the map ψ above by $\widehat{E(n)}^*(\mathbb{R}P^\infty)$; it is a subalgebra of $ER(n)^*(\mathbb{R}P^\infty)$. The E_1 page of the BSS is

$$E(n)^*(\mathbb{R}P^\infty) = E(n)^*[[u]]/([2]_F(u))$$

and $\widehat{E(n)}^*(\mathbb{R}P^\infty)$ maps to give us permanent cycles in $E_1^{*,*}$. In fact, these permanent cycles generate enough of the E_1-page such that knowledge of the differentials on the coefficients finishes the job. Concretely, let

$$\widehat{E(n)}^* = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, \widehat{v}_n^{\pm 1}]$$

where

$$\widehat{v}_i = v_i v_n^{(2^i-1)(2^n-1)}$$

is the result of applying ψ to $v_i \in E(n)^*$ and then mapping into $E(n)^*(\mathbb{R}P^\infty)$. Then it turns out that

$$E_1^{*,*} = E(n)^* \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty) = E_1^{*,*}(pt) \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty)$$

Note that, by construction, the right hand component of the tensor product consists entirely of permanent cycles. It thus turns out that the entire spectral sequence plays out on left hand component—that is, on the coefficients—where we know all of the differentials[a]. In particular, we thus have

$$E_\infty^{*,*} = E_\infty^{*,*}(pt) \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty)$$

and so

$$ER(n)^*(\mathbb{R}P^\infty) = ER(n)^* \otimes_{\widehat{E(n)^*}} \widehat{E(n)}^*(\mathbb{R}P^\infty)$$

Recall that $\widehat{E(n)}^*(\mathbb{R}P^\infty)$ is isomorphic to $E(n)^*(\mathbb{R}P^\infty)$. Thus, we see that $ER(n)^*(\mathbb{R}P^\infty)$ may be computed from $E(n)^*(\mathbb{R}P^\infty)$ simply by base change.

[a]The very astute reader may have noticed that this argument requires commuting homology past the tensor product at each stage of the spectral sequence. That this turns out to work for us is the result of a certain property, Landweber flatness (see [**KLW16a**] for more details).

1.6. The main theorem. The computation in our example generalizes to many different spaces. We have

THEOREM 2. *Let Z denote one of the Eilenberg MacLane spaces $K(\mathbb{Z}, 2q+1)$, $K(\mathbb{Z}/(2^k), 2q)$, or $K(\mathbb{Z}/(2), q)$ or one of the classifying spaces BO, BSO, $BSpin$, or $BString$ (the last for $n \leq 2$ only). Then $ER(n)^*(Z)$ may be computed from $E(n)^*(Z)$ by base change. That is,*

$$ER(n)^* \otimes_{\widehat{E(n)}^*} \widehat{E(n)^*}(Z) \longrightarrow ER(n)^*(Z)$$

where the subalgebra $\widehat{E(n)}^*(Z)$ is abstractly isomorphic to $E(n)^*(Z)$ after a suitable rescaling of degrees.

Each of the spaces Z above has a corresponding space Y with projective property that allows us to produce permanent cycles via the map ψ as above. The remaining ingredient is to show that these permanent cycles generate the E_1-page over the coefficients. In [**KLW16a**] this is formulated as a condition on the pair (Z, Y), leading to the definition of the titular Landweber flat Real pairs. Demonstrating the additional properties hold takes significant work, but it consists of classical nonequivariant computations that were fortunately already carried out for us in the beautiful papers [**RW80**, **RWY98**] and [**KLW04**].

2. Metaphors and imagery

2.1. Computations in algebraic topology. As described in the first part of this user's guide, a significant aspect of algebraic topology involves computing the values of cohomology theories on various topological spaces and studying the spaces by means of these algebraic invariants. Computations often proceed by the following metaphor:

METAPHOR 1. *Slice up the given object of interest into more manageable pieces, compute those pieces, then put them back together.*

If we are computing $E^*(X)$ for some cohomology theory E and some space X, this may apply to slicing up the space X, the theory E, or both. Examples of this notion of slicing things up into pieces include CW-decompositions, Postnikov towers, or iterating self-maps and taking quotients.

The challenge often comes in the stage of putting the basic slices back together. The methods by which topologists build things up (e.g. cofiber sequences) often induce *long* exact sequences upon applying algebraic invariants. If we built something up out of an iterated sequence of cofibrations, then in cohomology we have an iterated sequence of long exact sequences, and now the challenge is to figure out exactly how they fit together. This process is encoded in the notion of a spectral sequence. Spectral sequences are notoriously messy and complicated. Rather than get bogged down in the details, we will illustrate some ways of thinking about them by means of the example most relevant to [**KLW16a**].

2.2. The Bockstein spectral sequence. Recall from the first topic our two cohomology theories of interest, Johnson-Wilson theory, $E(n)$, and Real Johnson-Wilson theory, $ER(n)$, which fit into a cofiber sequence (first constructed in [**KW07a**])

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x} ER(n) \xrightarrow{p} E(n)$$

where $\lambda(n)$ is a certain positive integer and the first map is multiplication by a class $x \in \pi_{\lambda(n)} ER(n)$. This sequence relates a cohomology theory which we generally know how to compute, $E(n)$, to a cohomology theory we are interested in computing, $ER(n)$. We use the above sequence to build a tower:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{x} & ER(n) & \xrightarrow{x} & ER(n) & \xrightarrow{x} & ER(n) \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \cdots & & E(n) & & E(n) & & E(n) \end{array}$$

Note that we have now omitted the suspensions from our notation and we remember that the maps pictured may change degrees. We want to compute the $ER(n)$-cohomology of a space X, so we are going to apply the functor $[X, -]$ to the above tower. We now have an exact couple, i.e. something that looks like

$$\xymatrix{ \cdots \ar[r] & ER(n)^*(X) \ar[r]^x \ar[dl] & ER(n)^*(X) \ar[r]^x \ar[d]^p & ER(n)^*(X) \ar[d]^p \\ \cdots & E(n)^*(X) \ar[u]^\delta & E(n)^*(X) \ar[u]^\delta & }$$

where δ is the connecting homomorphism.

Let us consider how this structure gives us a method for computing $ER(n)^*(X)$ from $E(n)^*(X)$. The bottom row of the above diagram consists of the identical slices $E(n)^*(X)$, which we assume are known. We claim that $ER(n)^*(X)$, the object we want to compute, can be built up from these pieces by studying how the different copies of $E(n)^*(X)$ fit together.

Rather than thinking through this abstractly, I find it helpful to follow through what happens to actual elements of $ER(n)^*(X)$ and $E(n)^*(x)$ as we run the spectral sequence. Let's describe this by means of the following two thought experiments. The first concerns how classes in $ER(n)^*(X)$ get detected by the copies of $E(n)^*(X)$. The second concerns how certain classes in $E(n)^*(X)$ detect classes in $ER(n)^*(X)$.

Thought Experiment 1. Suppose we have a class z living in the right-most copy of $ER(n)^*(X)$ in the exact couple above. Our goal is to detect z in one of the copies of $E(n)^*(X)$. We may map z into the first $E(n)^*(X)$ along p. If $p(z)$ is nonzero, then we have detected z and we are done. If $p(z) = 0$, then by exactness, we must have that z is divisible by x, i.e. $z = xz_1$ for some z_1. That is, z comes from the copy of $ER(n)^*(X)$ second from the right and its lift in that copy of $ER(n)^*(X)$ is given by z_1. Now we play the same game with z_1. We map it into the second copy of $E(n)^*(X)$ along p. If its image is nonzero, then we have detected it in something we know. If the image is zero, then z_1 is divisible by x (so z was a multiple of x^2) and we lift z_1 to a z_2 in the third copy of $ER(n)^*(X)$. The claim is that eventually, this process ends and we see z detected in some copy of $ER(n)^*(X)$. Why? Because $x^{2^{n+1}-1} = 0$ so eventually the only way z can be in the image of $x^{2^{n+1}-1}$ is if $z = 0$. Thus, every element of $ER(n)^*(X)$ is detected in some copy of $E(n)^*(X)$.

This gives an argument that the things we care about, classes in $ER(n)^*(X)$, show up in the slices $E(n)^*(X)$. But there is more in our copies of $E(n)^*(X)$ than just elements of $ER(n)^*(X)$. The spectral sequence starts with the copies of $E(n)^*(X)$ and, step-by-step, filters out anything that is not in detecting something coming from $ER(n)^*(X)$. So let us do a second thought experiment to trace through how we determine whether a given class in $E(n)^*(X)$ is detecting an element of $ER(n)^*(X)$.

Thought Experiment 2. Suppose we have some $w \in E(n)^*(X)$ and we want to know whether it is in the image of $ER(n)^*(X)$. We want to study the image of w along δ because if $\delta(w) = 0$, then by exactness, w must come from $ER(n)^*(X)$. On the other hand, if $\delta(w) \neq 0$, then w does not come from $ER(n)^*(X)$ and so it is not useful to us. The difficulty is that we do not know much about $\delta(w)$ because it lives in $ER(n)^*(X)$ (the very thing we are trying to compute). But we can attempt to determine whether $\delta(w) = 0$ by mapping it into the copies of $E(n)^*(X)$. Let's begin by mapping $\delta(w)$ into the next copy of $E(n)^*(X)$ to the left along p. This composite, $p \circ \delta$, is the first differential, d_1. If $d_1(w) \neq 0$, then $\delta(w) \neq 0$ and we are done with our analysis (w is not useful to us and we throw it away by taking homology with respect to d_1 as w will fail to be in the kernel and will disappear). If $d_1(w) = 0$, however, then $\delta(w)$ has a chance of being zero and we continue the analysis. Our class w has survived to the E_2-page! Now $\delta(w) \in ER(n)^*(X)$ maps to zero in $E(n)^*(X)$ under p, so it must lift to the left to the next copy of $ER(n)^*(X)$. So we lift it and map down to the next copy of $E(n)^*(X)$ along p. We are now two copies of $E(n)^*(X)$ to the left of where we started. This is the definition of d_2. Again, if we get something nonzero, then w does not show up in homology with respect to d_2 and disappears on the E_3-page. Otherwise, if $d_2(w) = 0$, then w survives to the E_3-page and it may be lifted further up the tower. Eventually, if w survives all 2^{n+1} of the differentials, then $\delta(w)$ must be in the image of $x^{2^{n+1}-1} = 0$. Then $\delta(w) = 0$, which means w is in the image of the map from $ER(n)^*(X)$. That is, w detects some class z in $ER(n)^*(X)$.

I find running these two thought experiments to be the easiest way of explaining the inner workings of the Bockstein spectral sequence. But when it comes to computing it in practice, it is useful to have effective ways of actually organizing the data. To that end, we have the following metaphor:

Metaphor 2. *A spectral sequence is a book, each page of which is a grid. Computing a differential means we can flip the page. The goal is to compute the grid (or some part of the grid) on all of the pages, and the last page (if there is one) gives us the answer we want.*

In the case of the Bockstein spectral sequence, the first page just consists of copies of $E(n)^*(X)$ as the vertical lines of the grid. Since the rth copy is going to detect x^r-torsion in $ER(n)^*(X)$, we put in a formal variable, also called x, and index the vertical copies of $E(n)^*(X)$ by powers of x. To get from each page to the next, we compute a differential as described above. Each differential forces us to reckon with the next level of torsion. Since $x^{2^{n+1}-1} = 0$, the final differential is $d_{2^{n+1}-1}$ and $E_{2^{n+1}}^{*;*}$ is the last page of the book. From the last page, we may compute $ER(n)^*(X)$, as each vertical line of the last page represents the elements of the corresponding copy of $E(n)^*(X)$ which survived and represent x^r-torsion.[5]

In practice, it is hard to compute the entire spectral sequence from scratch starting with just the first page. More typically, we need to input various pieces of outside information. One way this can happen is if we know some *permanent cycles*. This means that some part of each page of the book is known to us through the entire book, and we can hopefully fill in the rest of each page from this data. This is the idea behind the hat construction described in Topic 1.

2.3. What does it mean to know a cohomology ring? If everything works out, the end result of a spectral sequence computation is knowledge of some algebraic object. But this turns out to be a surprisingly slippery issue in the context of [**KLW16a**] and many other computations in algebraic topology. The difficulty concerns what it means to 'know' a cohomology ring. One may take that to mean knowing the generators and relations. But in the case of the spaces considered in [**KLW16a**], we are dealing with quotients of power series rings. The relations between the generators are power series, which have infinitely many coefficients that must be 'known'. So what is the right way to think about these objects?

One sort of answer that occurs frequently in algebraic topology is a concise description of the power series which describe the relations between the generators. For example, the $E(n)$-cohomology of $\mathbb{RP}^\infty$ described in Topic 1 involved quotienting by the 2-series, $[2]_F(u)$. This power series is determined by the formal group law F for Johnson-Wilson theory. As we further unpack this, it becomes clear that the sense in which we know the 2-series is that we have an algorithm for computing as many terms of it as we like, even though we do not have a closed form description for its coefficients. This is the sort of knowledge we have of the $E(n)$-cohomology of all of the spaces of interest in [**KLW16a**].

[5]Note: the rth vertical line of the last page of the spectral sequence is actually giving us x^r-torsion modulo x^{r+1}-torsion, so the last step is to reconstruct $ER(n)^*(X)$ from this associated graded object. Weird things can happen (for example, we could theoretically have two x^3-torsion classes multiply together to give us an x-torsion class!) so there is an art to this final step (called solving extension problems).

Note that this is not the only possibly answer one could desire[6]. But these problems are just as relevant to $E(n)$-cohomology as they are to $ER(n)$-cohomology. In fact, the main result of [**KLW16a**] sidesteps this issue entirely by describing $ER(n)$-cohomology *in terms of* $E(n)$-cohomology for the spaces Z of interest:

$$ER(n)^*(Z) = ER(n)^* \otimes_{\widehat{E(n)}^*} \widehat{E(n)^*}(Z)$$

Thus, whatever the sense in which the $E(n)$-cohomology of the spaces of interest in [**KLW16a**] has been computed, we compute the $ER(n)$-cohomology in the same sense.

3. Story of the development

I will begin by outlining the history of Real Johnson-Wilson theory, $ER(n)$, before discussing how Nitu Kitchloo, W Stephen Wilson, and I came to be interested in the questions addressed in [**KLW16a**]. I will then give the specific history of [**KLW16a**], focusing on an approach that did not work, an approach that did, and what I learned about $ER(n)$ and computations from working on this project.

3.1. A brief history of $ER(n)$. Complex-oriented cohomology theories sometimes support C_2-actions, and this equivariance leads to both interesting structure and enhanced computational power. This observation traces back to Atiyah's work on Real K-theory [**Ati66**] and Araki, Fujii, Landweber, and Murayama's work on Real cobordism, $\mathbb{MU}$ [**Lan68**, **Fuj67**, **AM78**, **Ara79**]. In 2001, Hu and Kriz took the work of Araki and Landweber further by defining Real (meaning C_2-equivariant) analogs of BP, $E(n)$, and $K(n)$. Building on [**HK01**], Kitchloo and Wilson observed that the structure of the coefficients of $ER(n)$ makes it particularly amenable to computations [**KW07a**]. In the case of $ER(2)$, they used these computations in 2008 to prove many families of new non-immersion results for real projective spaces [**KW08a**, **KW08b**].

In 2009, Hill, Hopkins, and Ravenel demonstrated how using even more of the equivariance of complex-oriented theories (by studying the action of a larger group) yields deep new insights into both stable homotopy theory and geometry in their solution to the Kervaire invariant problem [**HHR09**]. Around the same time, $TMF_0(3)$, an equivariant cohomology theory built from modular forms and seemingly related to $ER(2)$ popped up in the work of Davis, Rezk, and Mahowald [**MR09**, **DM10**]. By the time I started working with $ER(n)$ in my second year of graduate school in 2011, equivariant perspectives on cohomology theories constructed from Real cobordism were developing from several directions.

[6] An example of this is given in [**Su06**]. Let $E(1,2) = E(2)/2$. It turns out that the $E(1,2)$-cohomology of $K(\mathbb{Z},3)$ is complete with respect to the topology given by powers of v_1 even though at no point did we apply v_1-completion. This is surprising and is not easily unpacked from the description of $E(1,2)^*(K(\mathbb{Z},3))$ as a quotient of a power series ring by an ideal whose generators may be computed to arbitrary degree by an algorithm.

3.2. How we came to this problem. The first research problem I attempted in graduate school was to determine which elements in the stable homotopy groups of spheres are detected in the coefficients of Real Johnson-Wilson theory. This seemed like a promising goal as $ER(1) = KO_{(2)}$ sees the class η and $ER(2)$ sees even more via the classes η, ν, and $\overline{\kappa}$. I hoped to find further spherical classes in the coefficients for $ER(n)$ for $n > 2$. After trying several different approaches to this problem, one of which seemed fruitful but petered out, my advisor Nitu Kitchloo and I began discussing a new approach via an $ER(n)$-based Adams-Novikov spectral sequence. The main obstruction to this program was knowing $ER(n)^*(ER(n))$, and we talked about computing $ER(n)^*(E(n))$ as an intermediate step. In one of these conversations, he pointed pointed out that even the $ER(n)$-cohomology of $\mathbb{C}P^\infty$ was not known and that it would be of independent interest. This became my thesis problem, published as [**Lor16**]. Sometime after I had worked out an approximate answer for $ER(n)^*(\mathbb{C}P^\infty)$ (the E_∞-page of the Bockstein spectral sequence), Nitu Kitchloo, W Stephen Wilson, and I began talking about extending this computation to other spaces.

The three of us first corresponded about the $ER(2)$-cohomology of $BU(q)$, but we quickly started talking more and more about approaching Eilenberg MacLane spaces as well (other than $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, which was the subject of my thesis). The space $K(\mathbb{Z}, 3)$ was the next logical choice to approach in this direction. Since $ER(n)^*(K(\mathbb{Z}, q))$ is trivial for $q > n+1$, this was the last integral Eilenberg MacLane space to compute for the theory $ER(2)$. We expected the computation to be at least as difficult as $\mathbb{C}P^\infty$ but were hopeful that it would not be intractable. The big surprise turned out to be that it is actually easier to compute! And more generally, it is easier to compute the $ER(n)$-cohomology of $K(\mathbb{Z}, \text{odd})$ than of $K(\mathbb{Z}, \text{even})$. As I will describe in the next section, it is entirely possible that expecting the computation to be difficult made us work through a lot of messy stuff that ended up being irrelevant. The previous year, Kitchloo and Wilson had computed the $ER(n)$-cohomology of BO using techniques that we generalize in [**KLW16a**]. The space BO rather than $\mathbb{C}P^\infty$ turned out to be the right analogy to focus on in our work on the $ER(n)$-cohomology of $K(\mathbb{Z}, 3)$.

3.3. How the results came about. Much of our early work on this problem happened over email, so I have a fairly extensive record of our progress. It is interesting to revisit the formation of our key ideas. In one of our first emails about $K(\mathbb{Z}, 3)$, we discussed how to think about $E(2)^*(K(\mathbb{Z}, 3))$ (the input for the BSS computing $ER(2)^*(K(\mathbb{Z}, 3))$) in terms of the sequence

$$K(\mathbb{Z}, 3) \longrightarrow BU\langle 6\rangle \longrightarrow BU\langle 4\rangle.$$

The pair $(K(\mathbb{Z}, 3), BU\langle 6\rangle)$ became our first new example of a Landweber flat real pair, and the above sequence of spaces is the first example that satisfies the conditions laid out in Section 5 of [**KLW16a**]. Later in the email chain, we briefly discussed the possibility of producing classes in the equivariant $E(2)$-cohomology of $BU\langle 6\rangle$ and then mapping them into the $ER(2)$-cohomology of $K(\mathbb{Z}, 3)$ to produce permanent cycles. This is exactly the method we ended up using in the paper. The ingredients we were missing early on were knowledge of the equivariant $E(2)$-cohomology of $BU\langle 6\rangle$ as well as the fact that the Bockstein spectral sequence could be written as a tensor product of the spectral sequence for the coefficients and an algebra of permanent cycles. The first of these two facts came from the projective property, which was a significant player in previous papers by Kitchloo and Wilson (beginning with [**KW07b**]). The second came from some remarkable left exact sequences of Hopf algebras from some great computational papers by Kitchloo, Wilson, and coauthors: [**RWY98**, **KLW04**].

Before we came around to the key ideas described in Topic 1, we spent much of the spring fiddling around with very concrete computations, especially involving the first differential in the Bockstein spectral sequence. Our hope was that the first differential would be all that we really needed to compute (as was the case, more or less, with $\mathbb{C}P^\infty$). Eventually, we started to converge on the fact that the value of the first differential on the 'hatted' power series generator of interest in $E(2)^*(K(\mathbb{Z},3))$ is zero. Around the same time, we had developed a long email chain in which we returned to the idea of using classes in the equivariant $E(2)$-cohomology of $BU\langle 6\rangle$ to produce permanent cycles. I remember a meeting in which we first recognized the argument that we had been circling around earlier of using the projective property to compute $\mathbb{E}(2)^\star(BU\langle 6\rangle)$. We also realized that this argument, together with sequences of the form

$$K(\mathbb{Z},2m+1) \longrightarrow \underline{BP\langle 2m-1\rangle}_{2(2^{2m}-1)} \longrightarrow \underline{BP\langle 2m-1\rangle}_{2^{2m}}$$

would allow us to generalize the result to all odd Eilenberg MacLane spaces. We had expected the computations for higher Eilenberg MacLane spaces to be more difficult, so this was an exciting development!

From here, the paper progressed quickly. That fall, motivated by a paper by Laures and Olbermann [**LO16**] in which they compute the $K(2)$-local $TMF_0(3)$ cohomology of $BO\langle 8\rangle$ (using techniques from [**KW14**]), we extended our results to some connective covers of BO. It was interesting to see the definition of Landweber flat real pairs apply to many more spaces than the ones that motivated it.

3.4. Lessons learned. As mentioned above, the computations in [**KLW16a**] turned out to be extremely clean. Much work would have been saved if we had known this ahead of time. As I see things now, there are two general sorts of methods to these $ER(n)$-computations: the nitty gritty approach, and the big picture approach. The nitty gritty approach consists of battling to compute each differential (often after filtering things first to simplify). The big picture approach consists of finding as many permanent cycles as possible (using some clever cofiber sequences or some geometry) and trying to generate a large part of the spectral sequence with them. The results of [**KLW16a**] turned out to be most susceptible to the latter. However, in parallel to [**KLW16a**], we also completed the computation of the $ER(2)$-cohomology of truncated projective spaces in [**KLW16b**]. This ended up being solved almost entirely by the nitty gritty method. I have come to see that both approaches are useful to making $ER(n)$ computations, and this fact is essential to its charm. There is just enough structure that one can compute and just enough chaos to make the results of the computations new and exciting.

4. Colloquial summary

The work carried out in [**KLW16a**] is part of a broad subfield of math called *algebraic topology.* At its essence, algebraic topology studies the rich and mysterious world of topology (a certain kind of geometry) by connecting it to the more tractable world of algebra. Before we discuss the work of interest to the rest of this user's guide, let's unpack this statement.

4.1. Geometry is more than what we see.

The sorts of shapes algebraic topologists study include any that you may care about. Some of them you see in the world around you: spheres, donuts, knots, or more complicated shapes like carburetors. But they also, importantly, include high dimensional shapes that we do not see with our eyes but which we can nevertheless describe abstractly.

QUESTION 4.1. *If we cannot see these shapes in the world around us, why should we care about them?*

The answer is that from a topologist's point of view, having a notion of 'nearness' is much more important than actually being able to see (either physically or in the mind's eye) the shapes of interest. Here are some examples of important high-dimensional geometric objects that never enter our visual field:

(a) the shape of all patient records of a given hospital[7],
(b) the shape of the voting preferences of all citizens in a given place,
(c) the shape of all configurations of n particles in space[8],
(d) the shape of all configurations of a robot with three arms of two joints each,
(e) the shape of all possible ways of packing bowling balls in a barrel,
(f) generally, the shape of all possible configurations of a system with quantifiable parameters.

For each of these examples we have some notion of what it means for two things to be near or far from each other. For example, we can talk about configuration A of a robot's arms being near configuration B if, overall, the arm placements in configuration A are near the arm placements in configuration B.

[7]The dimension is given by the number of parameters (e.g. age, height, weight, blood sugar) we keep for each patient.

[8]Example: the classical three-body problem takes place in 18-dimensional space. For each of three bodies, we need three dimensions to specify a position and three dimensions to specify a velocity.

This is, roughly, why we are justified in calling each of the above examples shapes even though we do not see them with our eyes.

The field of topology is specifically concerned with the *qualitative* aspects of these kinds of geometric objects. That is, a topologist is less interested in the precise distance between two points in a shape than in global features, such as whether there is a hole in the middle of the shape that separates one region from another. While interpreting what such a feature says about the actual physical situation is an important question[9], pure topology is concerned with understanding what sorts of global phenomena can occur, how they are related to each other, and how we can classify shapes based on these phenomena.

4.2. Cohomology theories. Algebraic topologists study the above sorts of shapes by coming up with various concrete and quantitative ways to measure their qualitative features. Some such ways of measuring are called *cohomology theories*, and they associate to each kind of shape a certain kind of very tractable and understood object. As a first approximation you can take our well-understood object to be a number.

Let's look at a concrete example. Suppose we have the following two shapes and we want to tell them apart.

One way to do this is to count the number of holes in each region and notice that we do not get the same numbers for each shape.[10]

This example illustrates that while shapes can be complicated, numbers are very concrete things. Algebraic topologists have invented many other ways of

[9]This is one aspect of the field of applied algebraic topology, which has found some striking use in the past decade to problems in fields as diverse as medicine, sensor networks, computer vision, and social choice theory, to name just a few.

[10]This example has much more complex analogs. Imagine the surface of a sphere. Now imagine the surface of a donut. These also have holes (i.e. the sphere has a 2-dimensional empty space inside of it), but of different natures from our shapes above. In higher dimensions, this gets even more complicated and the phenomena that occur become much more diverse and much stranger.

assigning numbers (or other rigid and well-understood objects like groups or vector spaces or rings) to geometric shapes that measure their various features. The results of [**KLW16a**] are concerned with one specific cohomology theory: *Real Johnson-Wilson theory.*

4.3. Real Johnson-Wilson theory and torsion. Real Johnson-Wilson theory has two advantages compared to other cohomology theories. The first is that the numbers it assigns shapes often exhibit a very special algebraic phenomenon called *torsion.* To get a sense of what torsion means, imagine the hours on a clock. The hour of day is a number but unlike the ordinary number line which keeps going indefinitely, once we reach the end of the (half) day at 12 hours, the hours start over. Torsion is the phenomenon of adding something to itself some number of times and getting back to the same place. An ordinary 12-hour clock exhibits 12-torsion. The numbers[11] produced by Real Johnson-Wilson theory are 2-torsion.

Many of the least understood phenomena in topology are closely connected with torsion. In fact, perhaps the biggest motivating problem in algebraic topology is computing the homotopy groups of spheres, which are algebraic objects that, roughly, contain all the information about all possible ways of constructing all possible shapes. The homotopy groups of spheres are almost entirely torsion, and they are in large part unknown.

The computations of [**KLW16a**] and research with Real Johnson-Wilson theory in general work because they detect only a small bit of torsion at a time, and they are constructed from a certain torsion-free cohomology theory (Johnson-Wilson theory[12]). Being torsion-free, computations with Johnson-Wilson theory are often susceptible to one of the most powerful tools in the mathematician's general toolbox: linear algebra. They are well-understood, and the results of [**KLW16a**] extend many known computations to the 'Real' context where there is interesting torsion to reckon with.

The computations in [**KLW16a**] are part of a larger program to study the way in which the torsion in Real Johnson-Wilson theory distorts some of the classical structure present in Johnson-Wilson theory. Besides being interesting for their own sake, these results sometimes yield applications to concrete geometric questions[13]. This is an important direction of future research. Our hope is that the results of [**KLW16a**] (as well as [**Lor16**, **KLW16b**] and work in progress) can be leveraged to produce some new geometric applications. In other words,

[11]more properly, groups

[12]Johnson-Wilson theory and Real Johnson-Wilson theory are *very different* cohomology theories. Real Johnson-Wilson theory should not be thought of as a version of Johnson-Wilson theory. Rather, it is constructed from Johnson-Wilson theory in a way that transmutes its structure in a controlled but significant way–in particular, it introduces torsion.

[13]For example nonimmersion problems, which have to do with when we can squeeze certain kinds of high-dimensional shapes into a high-dimensional Euclidean space.

now that we know more about the structure of Real Johnson-Wilson theory, we want to know what this tells us about the shapes that we care about.

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