Enchiridion: Mathematical User's Guides
`mathusersguides.com`
Volume 2 (September 2016)

A user's guide: An equivariant tensor product on Mackey functors

Kristen Mazur

Contents

1. Key insights and central organizing principles

1.1. Mackey functors are the new groups. Let G be a finite group. A goal of equivariant stable homotopy theory is to understand the homotopy groups of genuine G-spectra. Equivariant spectra are like spaces with a lot of extra structure. Hence, their homotopy "groups" also have a lot of extra structure. This extra structure stems from the fact that we want to define the k^{th}-stable homotopy group $\underline{\pi}_k$ of a G-spectrum X in such a way that it reflects not only the action of G on X but also the actions of all subgroups of G on X. So, $\underline{\pi}_k(X)$ is not merely a group. Instead, we define $\underline{\pi}_k(X)$ by first considering the group of homotopy classes of H-equivariant maps from the k^{th} sphere spectrum into X for all subgroups H of G. Then as H varies these abelian groups fit together to form a *Mackey functor*. We define $\underline{\pi}_k(X)$ to be this Mackey functor.

If X has *even more* structure, in particular, if X is a commutative G-ring spectrum, then in [**Bru07**] Brun shows that its zeroeth stable homotopy group is more than just a Mackey functor. A commutative G-ring spectrum comes equipped with norm maps, and we see the algebraic shadows of these norm maps in $\underline{\pi}_0(X)$. Hence, $\underline{\pi}_0(X)$ is a Mackey functor with commutative ring properties *and* internal norm maps that are like multiplicative transfer maps. It is a *Tambara functor*.

We want there to be a nice symmetry between G-spectra and their homotopy groups. So, we want the following idea to be true.

KEY IDEA 1.1. *Since Mackey functors are the new groups, Tambara functors should be the new rings.*

In other words, we want Tambara functors to play the role of rings in the category of Mackey functors. The category of G-Mackey functors $\mathcal{M}ack_G$ is a symmetric monoidal category, and so we can use the symmetric monoidal product and classic category theory to define ring objects (i.e. monoids with respect to the monoidal product) in $\mathcal{M}ack_G$. But alas! The ring objects under the symmetric monoidal product are merely Green functors. They are *not* Tambara functors. Green functors have less structure than Tambara functors because they do not have norm maps. Therefore, the goal of [**Maz16**] is as follows.

GOAL 1.2. *For G a cyclic p-group, develop an equivariant symmetric monoidal structure on $\mathcal{M}ack_G$ under which Tambara functors are the commutative ring objects.*

Hill and Hopkins developed an appropriate notion of equivariant symmetric monoidal, calling it *G-symmetric monoidal*. They call the ring objects under this structure *G-commutative monoids* [**HH13**]. Hill and Hopkins [**HH13**] and Hoyer [**Hoy14**] have independently defined G-symmetric monoidal structures on $\mathcal{M}ack_G$. Ullman [**Ull13**] then provides an algebraic description of Hill and Hopkins' structure. At the core of these structures are functors $N_H^G\colon \mathcal{M}ack_H \to \mathcal{M}ack_G$ for all subgroups H of G that send an H-Mackey functor to a G-Mackey functor. Hill and Hopkins and Ullman defined N_H^G by passing to an H-spectrum via the Eilenberg-MacLane functor, applying the Hill-Hopkins-Ravenel norm functor and then returning to G-Mackey functors via $\underline{\pi}_0$. Hoyer defined the functors N_H^G via coends.

In [**Maz16**], for G a cyclic p-group, we present a novel approach to building a G-symmetric monoidal structure on $\mathcal{M}ack_G$. Our structure does not involve spectra or coends. Instead, we develop a *concrete* G-symmetric monoidal structure by constructing new functors N_H^G using only the algebraic properties of Mackey functors and Tambara functors.

Hoyer shows that all of these G-symmetric monoidal structures are isomorphic [**Hoy14**]. However, there are still advantages and disadvantages to each. Hill and Hopkins', Ullman's and Hoyer's definitions of the functors N_H^G are easier to state, and it is easier to use these definitions to further develop concepts in equivariant stable homotopy theory. However, it is difficult to see how these functors affect any given Mackey functor. Furthermore, in order to really understand these definitions you need a good understanding of spectra. On the other hand, the definition of N_H^G in [**Maz16**] is nice because even though it is a bit messy, it does not rely on extensive knowledge of spectra or category theory, and it gives us a clearer understanding of what N_H^G does to individual Mackey functors. Thus, it is accessible to a wider range of mathematicians.

1.2. Discussion of the main construction. From here on let G be a cyclic p-group. Further, let $\mathscr{S}et_G^{Fin}$ be the category of finite G-sets. A G-symmetric monoidal structure on $\mathcal{M}ack_G$ is a functor

$$(-) \otimes (-) \colon \mathscr{S}et_G^{Fin} \times \mathcal{M}ack_G \to \mathcal{M}ack_G$$

that satisfies the properties given in Definition 5.1 of [**Maz16**]. In particular, $(-) \otimes (-)$ must break up disjoint unions of G-sets over the symmetric monoidal product $\square$ in $\mathcal{M}ack_G$. That is, given G-sets X and Y and G-Mackey functor $\underline{M}$,

$$(X \amalg Y) \otimes \underline{M} = (X \otimes \underline{M}) \square (Y \otimes \underline{M}).$$

Since every G-set can be written as a disjoint union of orbits, we can define $(-) \otimes (-)$ by defining $G/H \otimes \underline{M}$ for all orbits G/H and all G-Mackey functors $\underline{M}$. Hence, let $i_H^* \colon \mathcal{M}ack_G \to \mathcal{M}ack_H$ be the forgetful functor that sends a G-Mackey functor to its underlying H-Mackey functor. The following key idea completely determines our G-symmetric monoidal structure.

Key Idea 1.3. *For all G-Mackey functors $\underline{M}$ and all orbits G/H we define $G/H \otimes \underline{M}$ to be the composition $N_H^G i_H^* \underline{M}$.*

Thus, we define the G-Mackey functor $G/H \otimes \underline{M}$ by first reducing $\underline{M}$ down to an H-Mackey functor using i_H^* and then building a new G-Mackey functor via the functor $N_H^G \colon \mathcal{M}ack_H \to \mathcal{M}ack_G$. Indeed, the bulk of [**Maz16**] focuses on constructing the functors N_H^G for all subgroups H of G. The proof of Theorem 1.1 is not monumental because we meticulously developed N_H^G in Sections 3 and 4. Further, we define N_H^G so that our definition of the functor $(-)\otimes(-)$ maintains all of the properties of a G-symmetric monoidal structure and so that Tambara functors are the ring objects under this structure.

Key Idea 1.4. *All of the hard work lies in building the functors*

$$N_H^G \colon \mathcal{M}ack_H \to \mathcal{M}ack_G.$$

Let H be a subgroup of G, and let $\underline{M}$ be an H-Mackey functor. To define the functor N_H^G we first construct the G-Mackey functor $N_H^G \underline{M}$. We use the word "construct" deliberately. Definitions 3.2, 3.3, 3.4 and 3.9 of [**Maz16**] give detailed instructions for building each piece of $N_H^G \underline{M}$. We explicitly define every module $(N_H^G \underline{M})(G/K)$ and all restriction and transfer maps for all $K \leq G$. The collection of these definitions is long and looks intimidating, but it is actually fairly simple in the sense that it is just a detailed list of generators and relations. We then prove that the map $\underline{M} \mapsto N_H^G \underline{M}$ is a symmetric monoidal functor from $\mathcal{M}ack_H$ to $\mathcal{M}ack_G$ in Theorems 4.1 and 4.11.

So, how do we actually build the G-Mackey functor $N_H^G \underline{M}$? The construction is motivated by the fact that at the end of the day we want to use the collection of functors N_H^G for all subgroups H of G to create Tambara functors.[1]

Key Idea 1.5. *The G-Mackey functor $N_H^G \underline{M}$ must encode properties of a Tambara functor (especially the internal norm map N_H^G and Tambara reciprocity).*

We build $N_H^G \underline{M}$ from the ground up using the fact that all subgroups of G are nested. So, this is where we need the fact that G is a cyclic p-group C_{p^n}. We first define $(N_H^G \underline{M})(G/e)$. Then we use that module to define $(N_H^G \underline{M})(G/C_p)$ and continue up the ladder until we define $(N_H^G \underline{M})(G/G)$. We define the transfer and restriction maps so that they naturally satisfy all necessary properties of the corresponding maps of a Mackey functor. Moreover, the restriction maps satisfy properties analogous to the inherent properties of restriction maps in Tambara functors.

Further, to define $N_H^G \underline{M}$ we split this Mackey functor at the module $(N_H^G \underline{M})(G/H)$ and think about the chunk of $N_H^G \underline{M}$ that is below $(N_H^G \underline{M})(G/H)$ (i.e. $(N_H^G \underline{M})(G/H')$ for all subgroups H' of H) separately from the chunk above $(N_H^G \underline{M})(G/H)$.

The chunk below $(N_H^G \underline{M})(G/H)$:

If H' is a subgroup of H, then we define $N_H^G \underline{M}(G/H')$ to be $\underline{M}^{\Box |G/H|}(H/H')$. Further, the symmetric monoidal product is like a tensor product, and we can use this idea to get a better picture of what $\underline{M}^{\Box |G/H|}(H/H')$ looks like. Ideally, we would like $\underline{M}^{\Box |G/H|}(H/H')$ to simply be $\underline{M}(H/H')^{\otimes |G/H|}$. However, this module is not big enough to allow us to define the transfer map into $\underline{M}^{\Box |G/H|}(H/H')$ in a compatible way. Thus, letting H'' be the maximal subgroup of H', we define $\underline{M}^{\Box |G/H|}(H/H')$ to be a direct sum of $\underline{M}(H/H')^{\otimes |G/H|}$ and a copy of the module $(\underline{M}^{\Box |G/H|})(H/H'')$ modulo a Weyl action. We then define the transfer map $tr_{H''}^{H'}$ to be the canonical quotient map onto the latter summand. The two summands are connected by quotienting out by a submodule that creates relations analogous to Frobenius reciprocity. Hence, at the H' level, $N_H^G \underline{M}$ looks like the picture below.

[1] In Topic 2 we give more details and examples of $N_H^G \underline{M}$. We also give examples of ladder diagrams of Mackey functors and Tambara functors. If you have not yet seen these diagrams, it may be helpful to read Topic 2 before finishing Topic 1.

$$
\begin{array}{ll}
(N_H^G\underline{M})(G/H'): & \underline{M}^{\Box|G/H|}(H/H') = (\text{Tensor Part} \oplus \overbrace{\text{Image of Transfer Map}}^{(\underline{M}^{\Box|G/H|})(H/H'')/_{W_{H'}(H'')}})/_{FR} \\
 & \quad res^{H'}_{H''} \downarrow \quad \uparrow tr^{H'}_{H''} \\
(N_H^G\underline{M})(G/H''): & \underline{M}^{\Box|G/H|}(H/H'')
\end{array}
$$

The chunk above $(N_H^G\underline{M})(G/H)$:

If K is a subgroup such that $H < K \leq G$, then $(N_H^G\underline{M})(G/K)$ looks a lot scarier, but its general idea is the same. We want $(N_H^G\underline{M})(G/K)$ to feel like a part of a Tambara functor, but we need to create space for the image of the transfer map. So, this module is a direct sum involving a free summand generated by copies of $\underline{M}(H/H)$ and a transfer summand. The transfer summand is completely analogous to the image of the transfer map in the above diagram. The two summands are once again connected by quotienting out by the right submodule. However, this time we want the submodule to create relations like Tambara reciprocity. We will then be able to extract properties of a Tambara functor from the free summand. Below we showcase the K level of $N_H^G\underline{M}$ where K is the subgroup in which H is maximal.

$$
\begin{array}{ll}
(N_H^G\underline{M})(G/K): & (\text{Free Part} \oplus \overbrace{\text{Image of Transfer Map}}^{(N_H^G\underline{M})(G/H)/_{W_K(H)}})/_{TR} \\
 & \quad res^{K}_{H} \downarrow \quad \uparrow tr^{K}_{H} \\
(N_H^G\underline{M})(G/H): & \underline{M}^{\Box|G/H|}(H/H)
\end{array}
$$

In Section 4 of [**Maz16**] we develop all of the necessary properties of the functors N_H^G so that we can use them to create a G-symmetric monoidal structure on $\mathcal{M}ack_G$ under which Tambara functors are the G-commutative monoids.

Finally, in Section 5 we prove that our definition of $(-) \otimes (-)$ is in fact a G-symmetric monoidal structure on $\mathcal{M}ack_G$ and prove that Tambara functors are the G-commutative monoids. Most aspects of these two proofs are straightforward because we so carefully constructed the functors N_H^G. However, one of my favorite parts of this paper is showing that a G-commutative monoid is a G-Tambara functor. Indeed, if a Mackey functor $\underline{M}$ is a G-commutative monoid, then maps $X \to Y$ of G-sets induce maps $X \otimes \underline{M} \to Y \otimes \underline{M}$ of Mackey functors. To show that $\underline{M}$ is a Tambara functor we first show that $\underline{M}$ is a Green functor (i.e. a Tambara functor without the internal norm maps). This proof involves chasing diagrams of disjoint unions of single point sets. Then given a subgroup

H of G we define the internal norm map $N_H^G \colon \underline{M}(G/H) \to \underline{M}(G/G)$ by creating a composition

$$\underline{M}(G/H) \to N_H^G i_H^* \underline{M}(G/G) \to \underline{M}(G/G).$$

The first map of the composition follows from the fact that $\underline{M}(G/H)$ is isomorphic to $i_H^* \underline{M}(H/H)$ and so nicely maps into the free part of $N_H^G i_H^* \underline{M}(G/G)$. The second is part of the collection of homomorphisms that make up the morphism of Mackey functors $G/H \otimes \underline{M} \to G/G \otimes \underline{M}$. This composition possesses all properties of the internal norm map of a Tambara functor because of the way we defined $(N_H^G i_H^* \underline{M})(G/G)$.

Key Idea 1.6. *Under the G-symmetric monoidal structure that we created Tambara functors are the G-commutative monoids.*

2. Metaphors and imagery

For a cyclic p-group G, the backbone of the G-symmetric monoidal structure that we construct in [**Maz16**] is the collection of norm functors $N_H^G \colon \mathcal{M}ack_H \to \mathcal{M}ack_G$ for all subgroups H of G. These functors build a G-Mackey functor that looks like a Tambara functor using only the basic, algebraic properties of Mackey functors and Tambara functors. More specifically, given an H-Mackey functor $\underline{M}$, we build a G-Mackey functor $N_H^G \underline{M}$ that somehow encodes the properties of the norm maps of a Tambara functor. Then we can use $N_H^G \underline{M}$ to define these norm maps by creating a composition of homomorphisms that factors through $N_H^G \underline{M}$.

When creating the Mackey functor $N_H^G \underline{M}$ my goal was to build a "ladder diagram" of a Mackey functor that looks and acts like a Tambara functor. I spent a lot of time building and playing with these diagrams, especially for C_4-Mackey and Tambara functors. Moreover, I used the fixed point Tambara functor to help build intuition for the construction of $N_H^G \underline{M}$. Thus, here we will focus on constructing such ladder diagrams. We will specifically build diagrams for the fixed point C_4-Mackey functor and the fixed point C_4-Tambara functor. Finally, the general definition of the Mackey functor $N_H^G \underline{M}$ can be overwhelming. Thus, to demonstrate how to think about it we will build a ladder diagram for the C_4-Mackey functor $N_{C_2}^{C_4} \underline{M}$.

2.1. Mackey functors.

Throughout this discussion let G be a cyclic p-group. Formally, a G-Mackey functor consists of a pair of functors from the category of finite G-sets to the category of abelian groups that satisfy certain properties. But because of these properties, a G-Mackey functor $\underline{M}$ boils down to a collection of modules

$$\{\underline{M}(G/H) : H \leq G\}$$

along with maps between them. Since all subgroups of G are nested (because G is a cyclic p-group), we can imagine $\underline{M}$ as a ladder in which each rung is a module $\underline{M}(G/H)$. Then if H is a subgroup of K, there are two maps between

$\underline{M}(G/H)$ and $\underline{M}(G/K)$: the *transfer* map $tr_H^K\colon \underline{M}(G/H) \to \underline{M}(G/K)$ going up the ladder and the *restriction* map $res_H^K\colon \underline{M}(G/K) \to \underline{M}(G/H)$ going down the ladder. Moreover, if K' is a subgroup of G such that $H < K' < K$, then $res_H^K = res_H^{K'} res_{K'}^K$ and $tr_H^K = tr_{K'}^K tr_H^{K'}$, so we only need to determine tr_H^K and res_H^K when H is the maximal subgroup of K. Therefore, we picture Mackey functors in diagrams like Figure 1.

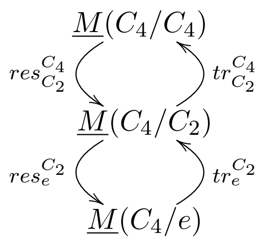
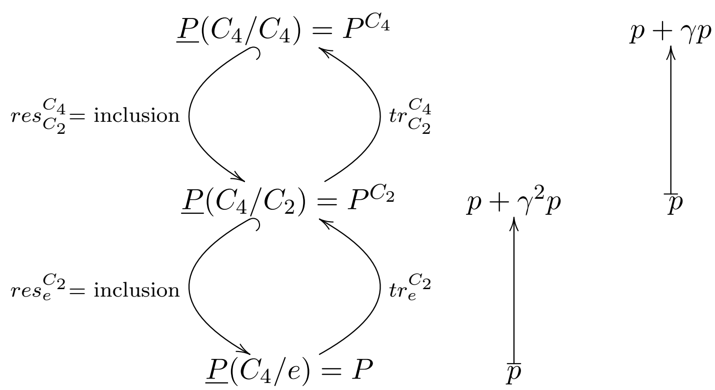


FIGURE 1. $\underline{M}$ is a C_4-Mackey Functor

Further, the restriction and transfer maps must play by certain rules. For example, the Weyl groups $W_K(H)$ act on each module $\underline{M}(G/H)$ whenever H is a subgroup of K. If γ is the generator of G and x is in $\underline{M}(G/H)$, then $res_H^K tr_H^K(x) = \sum_{\gamma^t \in W_K(H)} \gamma^t x$.

EXAMPLE 2.1. Let P be a C_4-module and let P^H denote the H fixed points of P for all subgroups H of C_4. Further, let γ be the generator of C_4. In the diagram below we have constructed the fixed point C_4-Mackey functor $\underline{P}$ from P. Notice that $tr_e^{C_4}(p) = p + \gamma p + \gamma^2 p + \gamma^3 p$ for all p in $\underline{P}(C_4/e)$.



2.2. Tambara functors. A Tambara functor is a Mackey functor with a lot of extra structure. So, to create a G-Tambara functor $\underline{S}$ we start with a Mackey functor ladder diagram and add a bunch of extra information to it. First, every $\underline{S}(G/H)$ is now a commutative ring instead of a module and the restriction maps become ring homomorphisms. (The transfer maps do not.) Then, whenever H is a subgroup of K we add another map $\underline{S}(G/H) \to \underline{S}(G/K)$ going up the ladder. This map is called a *norm* map and is denoted N_H^K. The norm maps are the multiplicative analogues of the transfer maps. For example, the norm maps are homomorphisms of multiplicative monoids (but are not additive), and $res^K_H N^K_H(x) = \prod_{\gamma^t \in W_K(H)} \gamma^t x$. A ladder diagram for a C_4-Tambara functor is given in Figure 2.

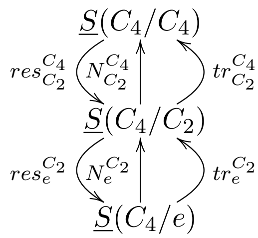


FIGURE 2. $\underline{S}$ is a C_4-Tambara Functor

Further, Tambara functors satisfy *Tambara reciprocity*, which tells us how the norm maps interact with sums and transfer terms. In particular, the norm maps are not additive, so, in general, $N^K_H(a+b) \neq N^K_H(a) + N^K_H(b)$, but via Tambara reciprocity, the norm maps are additive up to a transfer term. Hence,

$$N^K_H(a+b) = N^K_H(a) + N^K_H(b) + tr(-).$$

The specific make up of the transfer term $tr(-)$ depends on K and H and consists of sums of products of various Weyl conjugates of a and b. Similarly, if H' is a subgroup of H, there is a Tambara reciprocity formula that allows us to rewrite $N^K_H tr^H_{H'}(x)$ as the transfer $tr^K_{H'}$ of some element. Further, the Tambara reciprocity formulas depend only on the subgroups H', H and K of G. Every G-Tambara functor satisfies the same Tambara reciprocity properties. For example, in *every* C_{2^n}-Tambara functor,

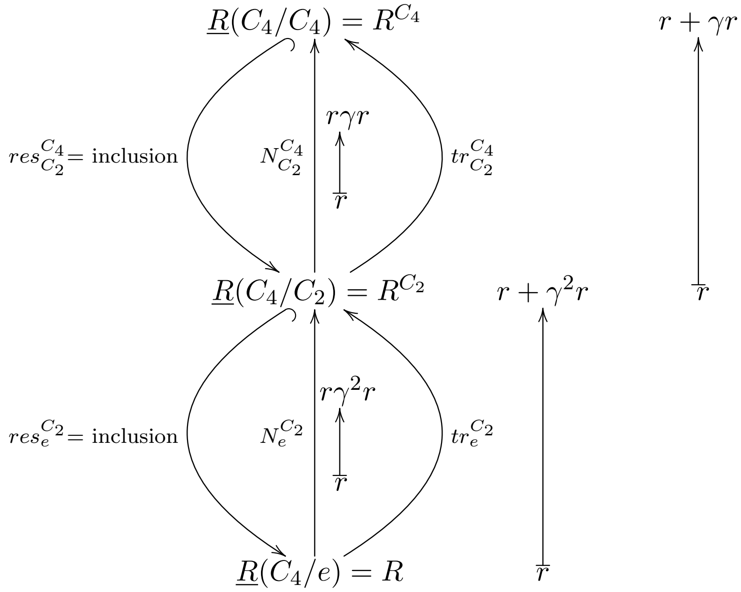
$$N^{C_2}_e(a+b) = N^{C_2}_e(a) + N^{C_2}_e(b) + tr^{C_2}_e(a\gamma^i b),$$

where γ^i is the generator of $W_{C_2}(e)$, and in *every* C_{2^n}-Tambara functor when $n \geq 2$,

$$N^{C_4}_{C_2} tr^{C_2}_e(x) = tr^{C_4}_e(x\gamma^j x),$$

where γ^j is the generator of $W_{C_4}(e)$.

EXAMPLE 2.2. We will build the fixed point C_4-Tambara functor $\underline{R}$ by adding structure to the fixed point C_4-Mackey functor in Example 2.1. First, start with a commutative C_4-ring R, instead of just a C_4-module. Then each $\underline{R}(C_4/H)$ is still R^H, and the restriction and transfer maps are as given in Example 2.1. Notice that the restriction maps are ring homomorphisms, but the transfer maps are not because they are not multiplicative. Adding the norm maps to the ladder diagram results in the picture below. Lastly, note that $N^{C_4}_e(r) = r(\gamma r)(\gamma^2 r)(\gamma^3 r)$ and that we can verify that $\underline{R}$ satisfies the Tambara reciprocity formulas mentioned above.



In [**Maz16**] we develop the formulas for Tambara reciprocity by chasing exponential diagrams. But these formulas are universally determined by the group G. This means that they will be the same in every G-Tambara functor. In particular, the formula for $N^G_H(a+b)$ in the G-fixed point Tambara functor is the same as the formula for $N^G_H(a+b)$ in *any other* G-Tambara functor. So, we can cheat and use the fixed point G-Tambara functor to come up with the appropriate formulas for the norm of a sum and the norm of a transfer in *any* G-Tambara functor. For example, in the C_4-fixed point Tambara functor above,

$$N^{C_4}_e(a+b) = (a+b)\gamma(a+b)\gamma^2(a+b)\gamma^3(a+b).$$

If we expand the righthand side of this equation, we can re-write it as

$$N^{C_4}_e(a) + N^{C_4}_e(b) + tr^{C_4}_{C_2}\left(N^{C_2}_e(a\gamma b)\right) + tr^{C_4}_e(a\gamma a\gamma^2 a\gamma^3 b + b\gamma b\gamma^2 b\gamma^3 a + a\gamma b\gamma^2 b\gamma^3 a).$$

Since this formula is universally determined by C_4, it will hold in every C_4-Tambara functor.

2.3. The Mackey functor $N^{C_4}_{C_2}\underline{M}$. Now, given an H-Mackey functor $\underline{M}$, we want to use these ladder diagrams of Mackey functors and Tambara functors to understand the G-Mackey functor $N^G_H\underline{M}$. But, when we try tackle this general case we quickly become bogged down in notation and details. So to simplify the exposition we will start with a C_2-Mackey functor $\underline{M}$ and build the ladder diagram for the C_4-Mackey functor $N^{C_4}_{C_2}\underline{M}$.

Recall that $N^{C_4}_{C_2}\underline{M}$ is a Mackey functor, but we want it to feel like a Tambara functor. Thus, let $\Box$ be the symmetric monoidal product (i.e. the box product) in the category of C_4-Mackey functors. We define $(N^{C_4}_{C_2}\underline{M})(C_4/e)$ to be $(\underline{M}\ \Box$

$\underline{M})(C_2/e)$ and $(N_{C_2}^{C_4}\underline{M})(C_4/C_2)$ to be $(\underline{M}\Box\underline{M})(C_2/C_2)$. The maps $res_e^{C_2}$ and $tr_e^{C_2}$ are the restriction and transfer maps of the box product definition (Definition 3.1 in [**Maz16**]). It remains to define $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$. As discussed in Part 1 of this User's Guide, $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$ consists of a free part and the image of the transfer map $tr_{C_2}^{C_4}$. We think of the free part as the home for norms, and so when we use this construction to create Tambara functors, we will pull norm elements from the free part. Thus, since the norms in a Tambara functor must satisfy Tambara reciprocity, we need to mimic this property in $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$. Hence, we quotient $(N_{C_2}^{C_4}\underline{M})(C_4/C_4)$ by the *Tambara reciprocity submodule* TR, and

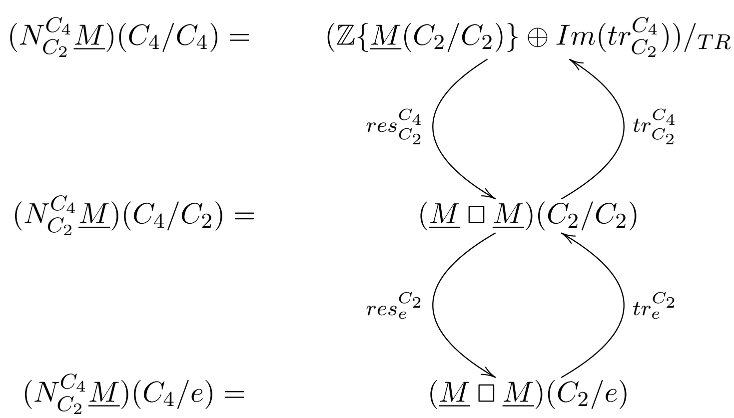
$$(N_{C_2}^{C_4}\underline{M})(C_4/C_4) = (\mathbb{Z}\{\underline{M}(C_2/C_2)\} \oplus Im(tr_e^{C_2}))/_{TR}.$$

We denote a generator of $\mathbb{Z}\{\underline{M}(C_2/C_2)\}$ by $N(a)$ for a in $\underline{M}(C_2/C_2)$, and so TR is generated by elements of the following forms for all a and b in $\underline{M}(C_2/C_2)$ and x in $\underline{M}(C_2/e)$:

$$N(a+b) - N(a) - N(b) - tr_{C_2}^{C_4}(a \otimes b)$$

$$N(tr_e^{C_2}(x)) - tr_e^{C_4}(x \otimes x).$$

These relations look familiar, right? We designed them so that they mirror the Tambara reciprocity relations that we saw in the discussion of Tambara functors. We visualize $N_{C_2}^{C_4}\underline{M}$ using the ladder diagram below.



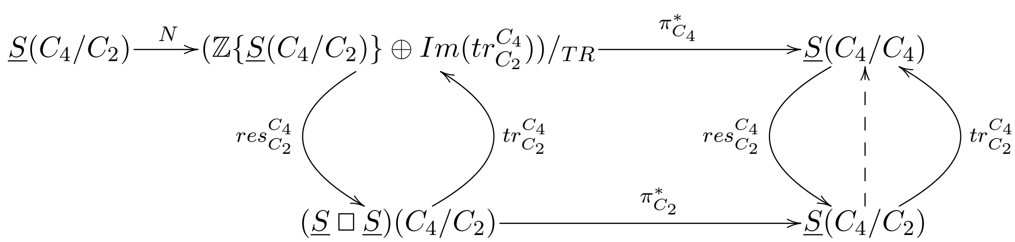
Finally, we will use this construction to define the internal norm maps of a Tambara functor. We first endow the category of G-Mackey functors with the G-symmetric monoidal structure defined in Theorem 5.2 of [**Maz16**]. So, by Proposition 5.8 of [**Maz16**] we know that if a G-Mackey functor $\underline{S}$ is a G-commutative monoid, then it has the extra structure of a Tambara functor. In particular, if $\underline{S}$ is a G-commutative monoid, then we can use the norm functors $N_H^G\colon \mathcal{M}ack_H \to \mathcal{M}ack_G$ to define the internal norm maps in $\underline{S}$. We will conclude by demonstrating how to build the internal norm map $N_{C_2}^{C_4}$ of a C_4-Tambara functor using the norm functor $N_{C_2}^{C_4}\colon \mathcal{M}ack_{C_2} \to \mathcal{M}ack_{C_4}$.

Let $\underline{S}$ be a C_4-Mackey functor that is a C_4-commutative monoid. Since $\underline{S}$ is a C_4-commutative monoid, we have a map

$$\pi^*\colon C_4/C_2 \otimes \underline{S} \to C_4/C_4 \otimes \underline{S}.$$

Recall from Part 1 of this guide that $C_4/C_2 \otimes \underline{S} = N_{C_2}^{C_4} i_{C_2}^* \underline{S}$ where $i_{C_2}^*$ is the forgetful functor, and so π^* is a morphism $N_{C_2}^{C_4} i_{C_2}^* \underline{S} \to \underline{S}$ of C_4-Mackey functors. Further, $(i_{C_2}^* \underline{S})(C_2/C_2)$ is isomorphic to $\underline{S}(C_4/C_2)$. Hence, we have a map $N\colon \underline{S}(C_4/C_2) \to (N_{C_2}^{C_4} i_{C_2}^* \underline{S})(C_4/C_4)$ that sends an element a in $\underline{S}(C_4/C_2)$ to the corresponding generator of the free summand of $(N_{C_2}^{C_4} i_{C_2}^* \underline{S})(C_4/C_4)$.

We use the ladder diagram below to define the internal norm map $N_{C_2}^{C_4}\colon \underline{S}(C_4/C_2) \to \underline{S}(C_4/C_4)$ of $\underline{S}$. This norm map is the dashed map on the right side of the diagram, and we define it to be the composition $\pi_{C_4}^* N$ that goes across the top of the diagram. This composition will satisfy all properties required of a norm map in a Tambara functor because of the way we constructed $N_{C_2}^{C_4} i_{C_2}^* \underline{S}$.



3. Story of the development

There are two phases to the development of a mathematical paper: the research phase and the writing phase. We sit and think during the research phase. We create pages of notes that mostly consist of wrong turns, doodles, and teardrops, but eventually the light bulb turns on and we develop new mathematics. We have conceived pages of definitions, theorems and proofs and are ready to share our findings with the mathematical community. Hence, we need to turn our notes into a cohesive and publishable document. We move into the writing phase. While the writing phase often becomes overshadowed by the research phase, I believe that developing a coherent story remains a critical process in mathematics research. Why discover new mathematical concepts but then obscure them beneath murky writing? As a result, here I would like to focus on the writing portion of this project.

I grew as a mathematician during the research phase of this project, but I *grew up* as a mathematician during the writing phase. I started to learn how to communicate to the math world, and I developed a deeper understanding of this project and its role in mathematics. This paper summarizes my graduate work at the University of Virginia. As a PhD candidate, I spent a lot of time tinkering with Mackey functors and Tambara functors, and working through examples and confusion. By early 2011 I established a cycle in which I would think for a week, ask my advisor questions, and then spend the next week understanding his answers and formulating new questions. This process continued for a few years, but eventually I made significant mathematical strides with Mackey functors and Tambara functors. I decided to graduate in May 2013. So, in the spring of 2013, with graduation looming, I needed to prepare a thesis for my defense. The writing phase began.

I did not write much during graduate school, and my thesis was the first formal mathematical document I created. As I wrote it I did not think about how to present my research. Instead I just wrote down absolutely everything that I discovered or learned about Mackey functors and Tambara functors. Moreover, I did not organize it in a logical way and did not develop proper mathematical notation for the story. For example, I devoted ten pages to the proof of what is now Theorem 4.7 in [**Maz16**] because I did not develop a practical way to describe generators and Weyl actions. As a result, these pages consisted of long strings of generators and relations that were impossible to read. (For instance, one page only contained 18 words, and I filled the rest of the page with symbols.) Regardless, I successfully defended my thesis in April 2013. However, while I earned my PhD, I felt dissatisfied with my thesis because I knew I could write a better story. Indeed, I had a significant amount of work ahead of me to turn the thesis into a publishable paper. However, I also needed to prepare for a visiting assistant professor position with a high teaching load. Thus began the three-year process of converting my thesis into a paper.

I worked on this paper on and off between the fall of 2013 and the fall of 2015. I started by developing a language and symbol system for telling my story in a reasonable number of pages. In particular, a crucial part of $N_H^G \underline{M}$ consists of large cartesian products and tensor products of $\underline{M}(G/H)$. I needed to carefully describe how each Weyl group acted on individual elements of these products. Hence, I developed notation that succinctly describes specific elements and how a generator of a Weyl group acts on these elements.

I also found a few mathematical glitches that I needed to fix. For example, I originally ignored the underlying G-action on $i_H^* \underline{M}$ when using $N_H^G i_H^* \underline{M}$ to define a G-symmetric monoidal structure on the category of Mackey functors. This was a mistake, and I had to figure out how this action fit into the story. I also spent a month in the summer of 2015 developing new and improved proofs of the formulas for the norm of a sum and the norm of a transfer in a Tambara functor (Theorems 2.3 and 2.4 in [**Maz16**]).

Most frustratingly, I learned that I had no idea how to write a cohesive and coherent mathematical paper. So I turned to the Internet. I googled "how to write a mathematical paper" and read many research papers, paying close attention to their wording and style. Through trial-and-error and emulation of the papers I read, I created a much-improved version of my paper and submitted it to a journal in August 2015.

Then on December 4, 2015 the second best thing for this paper happened: it was rejected. The editor said that the paper should be published but was not strong enough for the journal. I felt incredibly disappointed, but after the tears subsided I resolved to try again. I decided to fix the issues that the referee noted and resubmit the paper to another journal.

While working on the revisions, I gave a seminar talk at a nearby university. Here I met a topologist who agreed to read my paper and help me polish it up. This was the best thing to happen to my paper, and acts as a testament to the importance of verbal communication in the mathematical community. This topologist made me realize that my paper did not "tell a story." In particular, I organized the theorems and topics of the paper temporally, but this order lacked coherence to the reader. The topologist also suggested that I focus more on the norm functors that make my research unique and less on the creation of a G-symmetric monoidal structure that I originally stated as the Main Theorem. Moreover, I realized that I defined terms like the box product and G-commutative monoids way before I needed to use them, so readers did not immediately understand their importance to my project and did not remember what they were when these terms appeared later on. I also needed to break many of my proofs into several lemmas in order to frame a more readable paper. Overall, the structural organization of the paper made it difficult to distinguish between the story and the technical details.

I spent the spring of 2016 completely reorganizing the paper. First and foremost, I rearranged the paper into the following four sections:

- Tambara functors satisfy Tambara reciprocity
- Constructing the norm functors
- The norm functors satisfy the required properties
- Tambara functors are the G-commutative monoids.

With this new outline in mind I developed a main theorem that focuses on the construction of the norm functors, moved definitions to right before I actually use them, and broke long proofs up into digestible lemmas. I also rearranged content within each section and added more intuitive explanations to some confusing concepts.

Now that the re-organization is complete, I have submitted the paper to another journal. Although this process has been long and frustrating at times, through it I have grown as a writer and a mathematician. I learned how to fully develop a story from mathematical research, and I learned the importance of the writing phase in the evolution of a research project. Most importantly, I have created a paper that I am proud to share with my peers.

4. Colloquial summary

In the past if a lay-person asked me about this project or my field of research, my eyes would light up and the following Taylor Swift lyric would pop into my brain.

"I can show you incredible things
Magic, madness, heaven, sin." [**Swi14**]

I would eagerly explain how my goal is to learn about fancy, mathematical objects called G-spectra. However, because G-spectra prove difficult to study directly, we assign objects called Mackey functors to them, and we subsequently learn about G-spectra by studying Mackey functors. Unfortunately, my explanation inevitably caused most eyes to quickly glaze over no matter how enthusiastically I presented it. The technical aspects of algebraic topology and my project in particular make it difficult to appreciate their beauty. For example, it took me three years of graduate school in pure mathematics and fluency in a language of mathematics called category theory to even understand the *question* I hoped to answer. This project lives in such a small niche of mathematics that I cannot easily explain it to my colleagues at Elon because they do not research algebraic topology. If my professional colleagues struggle with the language and technicalities associated with studying G-spectra and Mackey functors, then I find it harder still to try to explain it to a general audience.

However, when I mention my career choice to general audiences they are usually intrigued by my life as a math professor. While not necessarily interested in the nuts and bolts of algebraic topology, they are often curious about my personal relationship with my field of research and about the impacts of my research on society. Questions about how I ended up in this field, why I find it interesting and what will come out of my research come up frequently in social engagements, yet still catch me off guard. I never really know how to answer these questions, and that is not only a disservice to my conversation partner, my career, and myself but also to the entire mathematical community. Good answers to such questions will help a general audience appreciate mathematics and mathematics research even when the research itself is too technical to follow. Thus, I have recently spent time pondering these questions and reflecting on why I chose a career in mathematics.

Therefore, I have divided this summary into two parts. The first is a very brief and general explanation of this project. It is aimed at an undergraduate student who has taken abstract algebra. My goal is to create a summary that is helpful for an undergraduate who is interested in going to graduate school in pure mathematics and wants to learn about a potential research area. In the second part of this summary I take the opportunity to share the answers that I have developed to three of the questions that I am most commonly asked when a stranger or non-academic acquaintance learns of my profession.

PART 1: TO THE CURIOUS UNDERGRADUATE

Either late in your undergraduate career or early in your graduate career you will take a course in topology in which you will learn about qualitative properties of mathematical objects called topological spaces. You already know some topological spaces such as the real number line, a sphere, a torus, or a mobius band. In graduate school you will be exposed to more topological spaces such as a klein bottle, the real projective space, manifolds, G-spaces, and spectra. It turns out that learning about advanced topological spaces is really hard. It becomes more difficult than abstract algebra. So, algebraic topologists have decided to use abstract algebra to learn about topological spaces. We realized that we could cleverly assign groups to spaces and gain a fair amount of information about spaces by studying these assigned groups instead of the spaces themselves. We call such groups 'invariants' of the spaces.

In graduate school I studied one type of invariant with the idea that if we can attain a better understanding of the algebra side of things then we can learn more about the topology side of things. But I did not study the groups that you learned about in abstract algebra. As topological spaces become more complicated plain old groups no longer suffice. We cannot gain enough information about the fancy topological spaces by assigning groups to them. We need invariants that encompass more information.

In the niche of equivariant stable homotopy theory one goal is to learn about topological spaces called G-spectra and their more elaborate siblings G-ring-spectra. One invariant for G-spectra is called a Mackey functor. The analogous invariant for G-ring-spectra is a Tambara functor. A Mackey functor is an incredibly robust group. If a group is a single gate at an airport, then a Mackey functor is an entire terminal. A Tambara functor is a Mackey functor with extra stuff added in, just as a ring is a group with extra stuff (notably the additional operation, multiplication).

I spent the bulk of my graduate career studying Mackey functors and Tambara functors. In truth I spent most of my time just trying to understand them by building examples and reading articles about them. But my end goal was to develop a new packaging for Mackey functors and Tambara functors that creates a new relationship between them. A better relationship between these algebraic objects will give us a better understanding of the relationship between the complicated topological things, G-spectra and G-ring spectra.

PART 2: COMMONLY ASKED QUESTIONS

I will next address three of the most popular queries that arise when I discuss mathematics and my research with someone without an advanced degree in the subject.

QUESTION 4.1. *What are the "real-life" applications for Mackey functors and Tambara functors?*

In truth, I do not yet know of any "real-life" applications of Mackey functors and Tambara functors, but I am also not currently interested in finding any. I like pure mathematics because I do not have to apply it to real-life. I like that I can escape to an abstract world and discover new mathematics without having to justify its existence. I can focus solely on the mathematics. However, though no real-life applications to my project currently exist, this may not always be the case. Mathematicians often develop theory and find an application for it later. For example, a major concept in algebraic topology is homology. Homology was developed in the late 1800s as a way to analyze mathematical shapes and spaces. But in the last decade we have adapted homology into a tool for analyzing big data. Perhaps we will discover some applications of Mackey functors and Tambara functors in the future, or perhaps studying them will lead to the development of other applicable mathematical concepts. Although my interest in Mackey and Tambara functors remains purely mathematical, I think it would be awesome if they helped create some new technology or played a role in the cure for a disease at some point in the future.

Question 4.2. *Why do you like researching such an abstract topic?*

I like math and specifically algebraic topology because I enjoy the mental challenge it presents to me. I love learning and growing by puzzling through difficult and abstract concepts. Working on this project felt like an intense training session for my brain. I dove into a dense world of functors, maps, actions and spectra. For months I continuously tried and failed to comprehend Mackey functors, Tambara functors, and how they best fit together. It was infuriating, but eventually I figured it out. Then I studied it some more until I could contribute to the field. That feeling of clarity and of finally understanding these objects was amazing. It made the struggle and frustration worth it!

Question 4.3. *What good will come out of your research?*

First, my research helps move the field of algebraic topology forward. It gives a new perspective on Mackey and Tambara functors, which is needed because Mackey and Tambara functors are particularly abstract mathematical objects. This new perspective will help topologists better understand these object which in turn will help us use Mackey and Tambara functors to create new mathematics.

But one of the more surprising outcomes of this project arose from the confidence and growth mindset that I developed from overcoming its challenges. Looking back on the time I spent studying Mackey and Tambara functors, I realize the value in all of the failed ideas and wrong turns that I took along the way. Of course these felt exasperating at the time, but they proved crucial to the research process. By learning what did not work and, more importantly, why it did not work, I gained a deeper understanding of the project. Moreover, knowing that I can understand and build abstract objects such as Mackey functors and Tambara functors granted me confidence in my ability to solve other difficult problems. Whenever I am stuck or frustrated I remind myself, "If you can understand Mackey and Tambara functors, then surely you can figure this out!" I have even applied this mentality to other research projects and academic challenges. For example, when I started working at Elon I had to teach a basic introductory statistics course that I never took in college. I needed to learn how to teach statistical concepts such as confidence intervals, hypothesis tests, and the Central Limit Theorem. The confidence and growth mindset that I gained through this research project were instrumental to my ability to overcome this challenge and successfully teaching the course.

In conclusion, the abstract and technical aspects of this project make it frustratingly difficult to convey its beauty to a general audience. But I hope that I have demonstrated why I find this research so provocative. It is a fun challenge that improves my ability to think critically and learn. Moreover, despite the lack of current real-life applications of Mackey functors and Tambara functors, the scientific community still benefits from studying them. Understanding G-spectra through Mackey and Tambara functors pushes the field forward and may one day provide a path towards future studies and applications. Indeed the contributions of this project both to my personal growth and to science and mathematics have proven to be subtle yet influential.

References

[Bru07] Morten Brun, *Witt vectors and equivariant ring spectra applied to cobordism*, Proc. London Math. Soc. **94** (2007), no. 2, 351–385.

[HH13] Michael A. Hill and Michael J. Hopkins, *Equivariant localizations and commutative rings*, 2013. in preparation.

[Hoy14] Rolf Hoyer, *Two topics in stable homotopy theory*, Ph.D. Thesis, 2014.

[Maz16] Kristen Mazur, *An equivariant tensor product on Mackey functors*, 2016. arXiv:1508.04062.

[Swi14] Taylor Swift, *Blank space, 1989*, Big Machine Records, 2014.

[Ull13] John Ullman, *Symmetric powers and norms of Mackey functors*, 2013. arXiv:1304.5648v2.

Department of Mathematics, Elon University, Elon, NC 27244

E-mail address: kmazur@elon.edu