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A user's guide: Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra

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Contents

1. Key insights and central organizing principles

1.1. Orientation/motivation. The paper [**Wol15a**] is a result of asking "modern" questions about "classical" stable homotopy theory. The majority of the proofs use calculations and facts that are specific to the setting of localized categories of spectra. Yet, the direction of inquiry is motivated by the larger setting of tensor-triangulated category theory. Let me explain.

As a subfield of topology, early stable homotopy theory (in the 1960s and 1970s) focused on computational results – of homology and cohomology groups, of maps between spectra, and of course of homotopy groups of spheres. In the 1980s there was a shift towards (or simply a birth of interest in) more global or structural questions about the category of spectra $\mathcal{S}$ and its localizations. Doug Ravenel proposed a list of conjectures in [**Rav84**] that helped establish and orient this line of inquiry. One of his conjectures was the original telescope conjecture (TC), which remains open. The other conjectures were answered, and in the process stable homotopy theory was formalized and axiomatized in a way that brought these structural questions further to the fore. The same questions could be asked in very different mathematical settings, like the derived category of a ring, or the stable module category of a group. (My thesis work was mainly about derived categories of non-Noetherian rings.) This is the setting of tensor-triangulated category theory, which finds common ground between topology, algebra, algebraic geometry, and representation theory.

In tensor-triangulated category theory we ask questions such as:

(1) What are the thick subcategories of compact objects?
(2) What are the localizing subcategories?
(3) What are the smashing localizations?
(4) What is the Bousfield lattice (if it exists)?

In the paper [**Wol15a**], I am mainly pursuing the last two of these four questions. The telescope conjecture is essentially a question about smashing localizations. One version is the Generalized Smashing Conjecture (GSC): Every smashing localization is generated by a set of compact objects. Another is the Strongly Dualizable Generalize Smashing Conjecture (SDGSC): Every smashing localization is generated by a set of strongly dualizable objects.

It was recently shown in [**IK13**] that every well-generated tensor-triangulated category has a set of Bousfield classes, hence a Bousfield lattice. In the hunt for "new" well-generated tensor triangulated categories to explore, I was drawn back to topology. The original TC in $\mathcal{S}$, despite many efforts, is open and thus hard. We know a lot about the Bousfield lattice of $\mathcal{S}$, but there are also many open questions that seem hard. One of the first key insights was to simplify the topological setting by localizing.

Key Idea 1.1. *Ask Questions 3 and 4 in localized categories of spectra.*

Localized spectra are a good setting because they are like spectra and unlike spectra, in interesting ways. Given a spectrum Z and homological localization $L_Z : \mathcal{S} \to \mathcal{S}$, with local category $\mathcal{L}_Z$, we can formulate a localized version of the TC. This is done in Section 3 of [**Wol15a**]. In non-topological categories, like derived categories of rings, this cannot be done (there is nothing like $\langle T(n)\rangle$ and $\langle K(n)\rangle$, or L_n and L_n^f), and one only has variations like the GSC and SDGSC to pursue (see [**Wol15a**] for precise definitions). Also, the close connection between $\mathcal{S}$ and $\mathcal{L}_Z$ allows us to compute in $\mathcal{L}_Z$ using what we know of $\mathcal{S}$.

On the other hand, $\mathcal{L}_Z$ can be quite unlike $\mathcal{S}$.

Key Idea 1.2. *If the localization L_Z is not smashing (which is most often the case), then the unit $L_Z S^0$ in $\mathcal{L}_Z$ is strongly dualizable but not compact.*

Thus in many cases $\mathcal{L}_Z$ is well-generated but not compactly-generated, which puts it in less-well-charted territory.

Local categories of spectra, especially the $K(n)$-local and $E(n)$-local categories, have certainly been studied for decades, but not often from the 'axiomatic' perspective. One notable exception is Hovey and Strickland's [**HS99**], which answers all four of the above questions for the $K(n)$-local and $E(n)$-local categories; their work was an inspiration for [**Wol15a**] and is summarized in Section 5 therein.

Broadly speaking, Section 3 formulates the questions and conjectures, and Sections 4–6 answer them in specific examples, setting Z to be $\bigvee_{i\geq 0} K(i)$, $K(n)$, $E(n)$, $H\mathbb{F}_p$, I, BP, or $F(n)$. I will discuss this in two subsections.

1.2. Questions and conjectures. To understand all the different versions of the telescope conjecture, it helps to lay them out chronologically. The observation that finite localization yields a smashing localization in $\mathcal{S}$ was first made by Bousfield in 1979, and that same year he also conjectured the converse, which is the GSC. Ravenel's telescope conjecture in [**Rav84**] is more specifically about the L_n and L_n^f functors, and holds if the GSC holds in $\mathcal{S}$. In the many attempts to prove Ravenel's conjecture, variations were developed. The conventional wisdom is that they are all equivalent, but this is only true if we quantify over all $n \geq 0$. For the sake of clarity, Definition 3.1 in [**Wol15a**] carefully distinguishes the main classical versions – $\mathsf{TC1_n}$, $\mathsf{TC2_n}$, $\mathsf{TC3_n}$ – of Ravenel's conjecture (which is $\mathsf{TC1_n}$), and Theorem 3.2 shows the (well-known) implications between them. This was necessary, since when we localize these statements (Definitions 3.6 and 3.11) they are not all equivalent all the time, as shown in Theorem 3.12, until we again quantify over all n.

The punchline here (see Theorem B), is that all the versions of the localized telescope conjecture all hold, for all $n \geq 0$, in all the examples I investigated. By shifting between variations and using Theorem 3.12 (as well as Bousfield lattice computations), it was not that hard to prove these results. So in the end I was unable to find a counterexample to the local telescope conjecture(s). Proposition 3.16 shows that a counterexample to the local telescope conjecture(s) would imply a counterexample to the original telescope conjecture(s), though, so perhaps this isn't surprising!

Besides these topology-specific localized telescope conjectures, we also have the GSC and SDGSC. The GSC is a great question to ask in any tensor-triangulated category, and has fueled research in interesting non-topological categories. One of the key insights of [**Wol15a**] is Theorem 3.5. I looked carefully at all the proofs that localization at a set of compact objects yields a smashing localization, and realized that by 'modernizing' the proofs I didn't need compactness, only strong dualizability. This gave birth to the SDGSC.

When the tensor unit is compact, strong dualizability and compactness are equivalent properties, so the SDGSC and GSC are the same thing. And many papers in tensor triangulated category theory are happy to assume the unit is compact. But localized categories of spectra, where the initial localization is not smashing, always yield tensor units that are not compact (and conversely, if the unit in the local category is not compact, then the localization is not smashing). So categories of local spectra are a perfect setting in which to tease apart the GSC and SDGSC. The punchline (see Theorem C) is that in several examples the GSC fails but the SDGSC holds.

While the SDGSC might be "the better thing" in this local setting, sadly it's necessary to point out that the SDGSC is also "not always the right thing". The only previous counterexample to the GSC, in Keller's [**Kel94**], is also a counterexample to the SDGSC. Keller looks at the derived category of a certain non-Noetherian ring, and in that setting the tensor unit is compact.

Finally, a few comments about the Bousfield lattice computations. They are directly relevant to the telescope conjecture, since $\mathsf{TC1_n}$ and $\mathsf{LTC1_n}$ are statements about Bousfield classes. Furthermore, since every smashing localization yields a pair of complemented classes, knowing something about the Bousfield lattice tells you something about the Boolean algebra of complemented classes, which tells you something about the smashing localizations.

1.3. The examples. The key insight in choosing examples was Corollary B.13 in [**HS99**], which says that there are no nonzero compact objects in the Z-local category for Z equal to any of the following: $\bigvee_{i\geq 0} K(i)$, $H\mathbb{F}_p$, I, BP or $H\mathbb{Z}$. These local categories are thus guaranteed to be well-generated but not compactly-generated. Furthermore, because there are no nonzero compact objects, any non-trivial smashing localizations on any of these local categories would imply failure of the GSC there. And this is exactly what happened (see Theorem C).

The case of $H\mathbb{Z}$ doesn't appear in the paper simply because I couldn't prove anything satisfying about it. I also include the cases of $K(n)$ and $E(n)$, reiterating the results of [**HS99**], mainly just for completeness. These two do provide compactly generated categories, in the first case because $L_{K(n)}F(n)$ is a compact generator and in the second because $L_{E(n)} = L_n$ is smashing.

Another reason for choosing these examples was that, except for BP, their Bousfield classes are "small" in the Bousfield lattice of $\mathcal{S}$. One of the key insights in relating Bousfield lattices of different categories is Lemma 2.8. It implies that if $\langle Z\rangle$ is small in $\mathsf{BL}(\mathcal{S})$, then $\mathsf{BL}(\mathcal{L}_Z)$ will be small. This is especially the case with $K(n)$, $H\mathbb{F}_p$, and I, and in fact this is how I prove that $\mathsf{BL}(\mathcal{L}_I)$ is the two-element lattice (Proposition 6.6).

In some ways, it seems like I chose the easy examples. But easy from a Bousfield lattice perspective certainly doesn't mean easy at the object level. One of the most fun results in the paper, Proposition 6.4, manages to take advantage of this discrepancy, and ties back to the hard question of classifying localizing subcategories.

2. Metaphors and imagery

In [**Wol15b**] I also described some of the ways that I think about Bousfield lattices, tensor-triangulated categories, and localization. What I say here is a supplement not a repetition.

2.1. Localization and local categories. Algebraic topology is algebraic and topological because we use functors between algebraic and topological categories. One of the most basic is homology. Given a spectrum Z in the stable homotopy category of spectra $\mathcal{S}$, the Z-homology is a functor Z_* from $\mathcal{S}$ to graded abelian groups. We translate topological questions into algebraic questions, and often the algebra is easier to work with. Working in the world of graded abelian groups, we're using Z as a tool to understand $\mathcal{S}$. The Z-local category of spectra $\mathcal{L}_Z$ is what we get if we "pull back" from the world of graded abelian groups to topology. But through this process we've lost information, or we've simplified $\mathcal{S}$, depending on how you look at it. Two spectra X and Y are identified as "the same" in $\mathcal{L}_Z$ if they have the same Z-homology. A map between spectra becomes an equivalence in $\mathcal{L}_Z$ if and only if it is an isomorphism in Z-homology. In this sense, $\mathcal{L}_Z$ is "what $\mathcal{S}$ looks like, according to Z".

The paper [**Wol15a**] can be confusing, because we are looking at localizations of localizations. For example, what are the smashing localizations in the harmonic localization of $\mathcal{S}$? (This is answered in Theorem 4.4.) I found it most useful to really inhabit a/the local category $\mathcal{L}_Z$ as much as possible. We can take the local category as the "base" category, somewhat forgetting that it came from $\mathcal{S}$. The axiomatic approach of tensor triangulated category theory – and in general any work in category theory – slowly develops an ability to shift contexts, to inhabit and relocate and re-inhabit different categories without losing your luggage or sense of direction. I think this is one of the joys of category theory papers and results.

What is interesting is to see how intuition seeps between categories. Understanding a category seems to involve knowing various structures and properties it holds, as well as a slew of random results about it, and proof and computational techniques for it. Shifting between categories, say from $\mathcal{S}$ to $\mathcal{L}_Z$, or from $\mathcal{S}$ to the derived category $D(R)$ of a ring R, one loses many but not all of these. At the beginning of working on [**Wol15a**], I felt that $\mathcal{S}$ had a lot of color, personality, and character, but a local category $\mathcal{L}_Z$ was uncomfortably unknown and vague. After all, taking $Z = S^0$ gives $\mathcal{L}_Z = \mathcal{S}$, and taking $Z = 0$ gives $\mathcal{L}_Z = 0$, so how much can one say? Maintaining this ambiguity, I was able to build understanding of $\mathcal{L}_Z$, but only formally.

For example, Lemma 2.10, about computations in $\mathcal{S}$, makes sense to me and evokes various imagery and perhaps even feelings. The objects $F(n)$, $T(n)$, and $K(n)$ have their own personalities and are like characters in a story. But Lemma 3.7, which proves the same computations in $\mathcal{L}_Z$, is completely symbolic and formal for me. It makes sense. But because Z can be anything from 0 to S^0, I can't really get comfortable with these objects $LF(n)$, $LT(n)$, and $LK(n)$, although I can compute with them.

This captures well the experience of doing category theory. It is like studying "language", and choosing a handful of specific languages to focus on as examples. You understand each language to a different degree. The ones that you understand well evoke rich memories and associations, and speaking them is natural and sensuous. When you understand a language poorly, the understanding is formal, discontinuous, more rational and less unconscious. And now imagine taking up a new language, which is somehow close or related to one that you know well – some intuition carries over to the new setting. But that only gets you so far, and can certainly be misleading.

Only towards the end of working on [**Wol15a**], when I was choosing specific Z, did I feel like the personalities of the $\mathcal{L}_Z$ were coming out. The $K(n)$-local category is boring from a tensor-triangulated perspective. In terms of vague intuition and a category theory perspective, the $E(n)$-local category is like $n+1$ copies of the $K(n)$-local category. The harmonic category is like the $E(n)$-local category, where n has been sent off to infinity. Strange things start to happen in the harmonic category though, making it much more interesting than the $E(n)$ case – e.g. localization is not smashing, and the localizing subcategories get out of hand. The $H\mathbb{F}_p$-local category is probably my favorite in [**Wol15a**], since $H\mathbb{F}_p$ is like the weird cousin who writes fantasy novels while his normal cousins the $K(n)$ are watching television. If I could live in a local category, it would be the $H\mathbb{F}_p$-local one. Or maybe the I-local category, which is also bizarre, but in a more pathological way, and perhaps a little too mysterious and hard to understand.

2.2. Telescope conjectures and the Bousfield lattice. A lattice, vaguely speaking, is a nice partially ordered set. The Bousfield lattice is a lattice extracted from any well-generated tensor-triangulated category: each object X gives a Bousfield class $\langle X\rangle = \{W \mid X \wedge W = 0\}$, and these form a lattice. In the context of [**Wol15a**], I found it useful to keep in mind the relationship between the Bousfield lattice and homological localizations. Given two spectra X and Y in $\mathcal{S}$, we have $\langle X\rangle \leq \langle Y\rangle$ in $\mathsf{BL}(\mathcal{S})$ if and only if L_X-locals are L_Y-locals. Of course this is the case if and only if L_Y-acyclics are L_X-acyclics (i.e. if $W \wedge Y = 0$ implies $W \wedge X = 0$), but for intuition I find the first characterization to be the better way to think about it. For computations, the second characterization is usually the more useful, but not always.

By definition, $\langle X\rangle = \langle Y\rangle$ if and only if L_X and L_Y are the same functor. The Bousfield lattice of a category T is a description of the different homological localizations that one can do on T. And this 'description' happens to have a lot of structure – e.g. it is a complete lattice, with additional operations like $- \wedge -$ and complementation.

I think of the telescope conjecture and all its variations, as described in Topic 1, as a wonderfully confusing mathematical story with lots of history. The original conjecture is hard and I have no hope of solving it. Now there are multiple versions, in different contexts. Many different people have very different perspectives and intuition about it. In my opinion, no one can really be said to *understand* the telescope conjecture, only different aspects of it.

The GSC and SDGSC are a little easier to grasp than the original topological versions $\mathsf{TC1_n}$, $\mathsf{TC2_n}$, and $\mathsf{TC3_n}$. They have a much more categorical feel. When I think of the GSC, I see a Venn diagram of a circle in a square. Every set of compact objects yields a smashing localization. It's like "every A is a B". The GSC asks if every smashing localization comes from a set of compact objects – "is every B an A?" What the SDGSC does is shift A to a slightly different A'. Theorem 3.5 says that every A' is a B, and the SDGSC asks if every B is an A'. But the relationship between A and A' – between compactness and strong dualizability – is category-dependent (see for example [**HPS97**, Theorem 2.1.3]). This is not a tightening of the GSC, but I guess I would call it an improvement, except for all those settings where the tensor unit is compact, in which case the GSC and SDGSC are equivalent, and A and A' are the same.

3. Story of the development

Careful background reading set the foundation. After reading [**IK13**], I became excited to look around for 'new' well generated tensor triangulated categories, in order to calculate their Bousfield lattices. Another result in [**IK13**], Proposition 2.1, points out that every Bousfield class $\langle Z\rangle$ of a well generated T is a well generated localizing subcategory. This implies that the Verdier quotient $\mathsf{T}/\langle Z\rangle$, which is equivalent to the local category $\mathcal{L}_Z$, is well generated. I found out about this early in fall 2012, and so began thinking about localized categories and their Bousfield lattices. I was in the mood to get back to topology, after writing a thesis on mostly homological algebra, so I started reading about localized categories of spectra.

I began this background reading in the first months of my postdoc position in fall 2012. Having earned my PhD in June in Seattle, and spent the summer traveling, I started a postdoc with Dan Christensen at the University of Western Ontario. However, Dan had a sabbatical year and so we spent the year as visitors at the Instituto Superior Técnico in Lisbon, Portugal. From September to December I worked through many different papers, planting seeds and looking for inspiration and good questions. A visit to the University of Bielefeld, to talk with Greg Stevenson about the harmonic category and with Henning Krause about the generalized smashing conjecture, was an important part of this. The cafés in Lisbon provided a great atmosphere for research, because not understanding Portuguese I was free from the distractions of written signs and overheard conversations. Moving alone to a new city in a new country provided lots of inspiration and free time.

Hovey and Strickland's paper [**HS99**] already answered all the questions I was interested in for the $K(n)$-local and $E(n)$-local categories. But studying their proofs gave me some ideas for other categories. Earlier in 2012, Jon Beardsley showed me that he had calculated the Bousfield lattice of the harmonic category. I spent some time trying to classify the localizing subcategories of the harmonic category, but gave up.

During this time I was also reading about the "telescope conjecture" – really the GSC – in various algebraic categories. I started to become curious about the whole telescope conjecture story.

In January and February 2013 I was distracted working on other research projects, but this allowed me to gain some perspective. In March and April 2013, while spending weekends riding my motorcycle to the Portuguese beaches and cliffs to surf or rock climb, I dove into the history of all (or most of) the variations of the telescope conjecture. By early April I had figured out the right questions to ask, such as: what are the different versions of the TC, in local categories of spectra? What are the implications between them? What are the Bousfield lattices of these categories? They seem obvious to me in hindsight, but formulating these questions – realizing that they are interesting and probably answerable – was more than half the total work. It became clear that I most likely wouldn't be able to answer Ravenel's original conjecture $\mathsf{TC1_n}$ in $\mathcal{S}$. It also became clear that the variations had non-trivial implications between them, which might create subtleties in the local setting. Around this time, and heading into May, I formulated the different variations $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, and $\mathsf{LTC3_n}$ for localized categories of spectra. But it wasn't until later in the summer, in southern France, that I dug the proofs in Theorem 3.2 out of the literature, and tweaked them to give the implications in Theorem 3.12 and 3.13.

Starting in late April, I headed off from Lisbon on a Europe motorcycle trip. I spent a week at the University of Seville talking with Fernando Muro. I rode up to Madrid and flew to the University of Copenhagen for ten days, then flew back and continued north to Barcelona. I visited Carles Casacuberta there for a week in late May.

At each university I would give a seminar or conference talk, and I regularly video conferenced with my postdoc supervisor in Lisbon. Throughout the trip I was working in offices or cafés on my research. In between each mathematical visit, I spent one or two weeks rock climbing and camping – at El Chorro, Rodellar, and Siurana. From Barcelona I went to southern France, where I climbed in the Gorges du Verdon for ten days. This naturally inspiring, simple lifestyle was very conducive to immersive thought. It was during this time, in mid-June, that I finalized most of the results and wrote most of the paper, sitting at a booth in the village bar, after mornings spent climbing.

An exciting turning point occurred in mid-May in Spain. I was reading through all the different proofs from the last three decades that localization at a set of compact objects yields a smashing localization. By piecing together different proofs, and using the new results in [**IK13**], I realized that I didn't need compactness and could prove it with a set of strongly dualizable objects. This gave birth to the SDGSC. I already knew that the GSC failed in the harmonic category; I had shown this the previous fall. Now I had hope that maybe the SDGSC would hold where the GSC failed. Over the next few weeks I showed that this is the case in the examples I was considering.

Having made some progress, in early June I pulled together a list titled "big results I hope for". This served as a roadmap.

As mentioned above, my muse was kind to me during ten days in southern France. I believe the (mathematical) catalyst was having the implications of Theorems 3.2, 3.12, and 3.13 worked out, and then turning to the range of examples to try to answer the localized telescope conjectures. I also began the process of writing up my results, and so I pulled together many classical computations into one spot, Lemma 2.10. With the clarity of Lemma 2.10, and its local version Lemma 3.7, and by moving between the versions – $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, $\mathsf{LTC3_n}$, GSC, SDGSC – and between all the different examples – harmonic, $H\mathbb{F}_p$-local, I-local, etc. – I was able to tease out all the results of Section 6 without getting caught up anywhere for too long.

Also in southern France, Proposition 3.8 and Lemmas 3.9 and 3.10 emerged concurrently and quickly. After months of having a vague sense that something was going on (e.g. that l_n^f might always be smashing, or that perhaps $l_n = LL_n$ in some cases), the statements and proofs of all three of these results came together in a day or two. One of the last things I proved – although perhaps one of the more conceptually important – was Proposition 3.16. This wasn't necessarily because it was hard. Maybe a part of me didn't want to know Proposition 3.16, because it says that it would be very unlikely to find a counterexample to one of the localized telescope conjectures.

After this binge of work, I took a week break to ride my motorcycle through the Italian and Swiss Alps. Unable to not think about the paper, though, I ended up continuing to write up the results, which led to thinking more about how to expand or improve what I had. After all, it wasn't clear to me where to stop. Most of the things on my list of "big results I hope for" were settled, but some refused to yield. After trying a bit more to no avail, I left some of them in the paper as Questions, and left some in my personal notebooks as future questions to pursue. From this point, all that remained was to write up the Preliminaries and Introduction, and submit. The final touches were done while attending the Young Topologist's Meeting in Lausanne in early July 2013. After riding up to Copenhagen to collaborate on a math-art project, and then to Berlin, I let the paper sit for a week or two and then submitted it from Berlin at the end of July.

Several months after submission, I returned to look at some of the unanswered questions. I had calculated the Bousfield lattice of the $H\mathbb{F}_p$-local category to be a two-element lattice, but had a feeling that the $H\mathbb{F}_p$-local category should have proper non-zero localizing subcategories. If I could find one, it would be a localizing subcategory that wasn't a Bousfield class, and this would be exciting and interesting. In December 2013 I posted a question to this effect on MathOverflow, and Mark Hovey quickly answered it, giving a proper non-zero localizing subcategory in the $H\mathbb{F}_p$-local category. This improvement (which I wasn't able to build on very much) became Proposition 6.4 in the final version of [**Wol15a**].

4. Colloquial summary

A "category" is a collection of mathematical objects, with "morphisms" – functions, or relationships – between them. It's a way of describing different mathematical "worlds" in which to work. Topologists work in categories where the objects are different types of spaces. Stable homotopy theory is a type of topology, where we study the category of "spectra", denoted $\mathcal{S}$. A spectrum is a strange sort of generalization of what we normally think of as space (one that allows for "negative-dimensional space", but that's another story...). The paper under discussion, [**Wol15a**], is a paper about stable homotopy theory, about spectra.

Rather than consider the usual category of spectra, I do something to it – I "localize" it. For each spectrum, call it Z, it is possible to localize $\mathcal{S}$ at Z, and we get the "Z-local category of spectra", denoted $\mathcal{L}_Z$. Intuitively, this category is "what $\mathcal{S}$ looks like to Z". The intuition behind localization agrees with the everyday notion: it's a way of simplifying $\mathcal{S}$, of focusing on particular aspects of it. In [**Wol15a**], I consider a bunch of different local categories: the BP-local category, the $H\mathbb{F}_p$-local category, the I-local category, etc. Each of BP, $H\mathbb{F}_p$, and I are different spectra, and their local categories are quite different. Each has special properties and characteristics, and each can tell us something different about the larger, un-localized category of spectra $\mathcal{S}$.

So in the paper I'm traveling between local categories of spectra, but what particular questions am I asking in each local setting? There are two main ones, related to each other.

One is computational. Each of these local categories of spectra has a "Bousfield lattice" associated to it. I want to know what it is. It's like saying, "Every town in Fox county has a ZIP code. What are these ZIP codes?" In all the examples I consider in the paper, I'm able to calculate the Bousfield lattice completely (except in one case, where all I can do is say something about how big it is). This leads to some interesting consequences (for example, Proposition 6.4: "In the $H\mathbb{F}_p$-local category, there is a localizing subcategory that is not a Bousfield class").

The second type of question is more elaborate. In a famous paper in 1984, the stable homotopy theorist Doug Ravenel announced a list of conjectures about spectra – good, hard questions worth pursuing. In the 1980s and 1990s all of them were answered, by various people, except one: the "telescope conjecture".

Roughly speaking, what is the telescope conjecture? It's a question about "smashing localizations". A smashing localization is an especially nice type of localizing – it results in a local category that is particularly nice in relationship to the larger category. The telescope conjecture aims to classify these nice localizations.

Rather than tackle the original telescope conjecture, a question in $\mathcal{S}$, my paper translates this telescope conjecture into the local categories, and asks it there. Sure enough, the localized telescope conjecture is easier to answer. Much of my paper is devoted to this purpose. But things have gotten complicated since 1984. Instead of one telescope conjecture, there are now four – all slightly different versions of the original. And to make matters worse, I invent a new one.

I'm able to answer all five versions of the conjecture in all the local categories I consider. For four of the five conjectures (including the one I invented), the answer is: yes, the conjecture holds, in all the categories considered. And I use my Bousfield lattice calculations in key ways to do this. However, one of the conjectures (the "GSC") fails, in several of the local categories I consider, and this is also interesting.

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