Enchiridion: Mathematical User's Guides Volume 2 (September 2016)
mathusersguides.com

A user's guide: Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra

F. Luke Wolcott

1. Key insights and central organizing principles

1.1. Orientation/motivation. The paper [**Wol15**] is a result of asking "modern" questions about "classical" stable homotopy theory. The majority of the proofs use calculations and facts that are specific to the setting of localized categories of spectra. Yet, the direction of inquiry is motivated by the larger setting of tensor-triangulated category theory. Let me explain.

As a subfield of topology, early stable homotopy theory (in the 1960s and 1970s) focused on computational results – of homology and cohomology groups, of maps between spectra, and of course of homotopy groups of spheres. In the 1980s there was a shift towards (or simply a birth of interest in) more global or structural questions about the category of spectra $\mathcal{S}$ and its localizations. Doug Ravenel proposed a list of conjectures in [**Rav84**] that helped establish and orient this line of inquiry. One of his conjectures was the original telescope conjecture (TC), which remains open. The other conjectures were answered, and in the process stable homotopy theory was formalized and axiomatized in a way that brought these structural questions further to the fore. The same questions could be asked in very different mathematical settings, like the derived category of a ring, or the stable module category of a group. (My thesis work was mainly about derived categories of non-Noetherian rings.) This is the setting of tensor-triangulated category theory, which finds common ground between topology, algebra, algebraic geometry, and representation theory.

In tensor-triangulated category theory we ask questions such as:

(1) What are the thick subcategories of compact objects?
(2) What are the localizing subcategories?
(3) What are the smashing localizations?
(4) What is the Bousfield lattice (if it exists)?

In the paper [**Wol15**], I am mainly pursuing the last two of these four questions. The telescope conjecture is essentially a question about smashing localizations. One version is the Generalized Smashing Conjecture (GSC): Every smashing localization is generated by a set of compact objects. Another is the Strongly Dualizable Generalize Smashing Conjecture (SDGSC): Every smashing localization is generated by a set of strongly dualizable objects.

It was recently shown in [**IK13**] that every well-generated tensor-triangulated category has a set of Bousfield classes, hence a Bousfield lattice. In the hunt for "new" well-generated tensor triangulated categories to explore, I was drawn back to topology. The original TC in $\mathcal{S}$, despite many efforts, is open and thus hard. We know a lot about the Bousfield lattice of $\mathcal{S}$, but there are also many open questions that seem hard. One of the first key insights was to simplify the topological setting by localizing.

Key Idea 1.1. *Ask Questions 3 and 4 in localized categories of spectra.*

Localized spectra are a good setting because they are like spectra and unlike spectra, in interesting ways. Given a spectrum Z and homological localization $L_Z : \mathcal{S} \to \mathcal{S}$, with local category $\mathcal{L}_Z$, we can formulate a localized version of the TC. This is done in Section 3 of [**Wol15**]. In non-topological categories, like derived categories of rings, this cannot be done (there is nothing like $\langle T(n) \rangle$ and $\langle K(n) \rangle$, or L_n and L_n^f), and one only has variations like the GSC and SDGSC to pursue (see [**Wol15**] for precise definitions). Also, the close connection between $\mathcal{S}$ and $\mathcal{L}_Z$ allows us to compute in $\mathcal{L}_Z$ using what we know of $\mathcal{S}$.

On the other hand, $\mathcal{L}_Z$ can be quite unlike $\mathcal{S}$.

Key Idea 1.2. *If the localization L_Z is not smashing (which is most often the case), then the unit $L_Z S^0$ in $\mathcal{L}_Z$ is strongly dualizable but not compact.*

Thus in many cases $\mathcal{L}_Z$ is well-generated but not compactly-generated, which puts it in less-well-charted territory.

Local categories of spectra, especially the $K(n)$-local and $E(n)$-local categories, have certainly been studied for decades, but not often from the 'axiomatic' perspective. One notable exception is Hovey and Strickland's [**HS99**], which answers all four of the above questions for the $K(n)$-local and $E(n)$-local categories; their work was an inspiration for [**Wol15**] and is summarized in Section 5 therein.

Broadly speaking, Section 3 formulates the questions and conjectures, and Sections 4–6 answer them in specific examples, setting Z to be $\bigvee_{i\geq 0} K(i)$, $K(n)$, $E(n)$, $H\mathbb{F}_p$, I, BP, or $F(n)$. I will discuss this in two subsections.

1.2. Questions and conjectures. To understand all the different versions of the telescope conjecture, it helps to lay them out chronologically. The observation that finite localization yields a smashing localization in $\mathcal{S}$ was first made by Bousfield in 1979, and that same year he also conjectured the converse, which is the GSC. Ravenel's telescope conjecture in [**Rav84**] is more specifically about the L_n and L_n^f functors, and holds if the GSC holds in $\mathcal{S}$. In the many attempts to prove Ravenel's conjecture, variations were developed. The conventional wisdom is that they are all equivalent, but this is only true if we quantify over all $n \geq 0$. For the sake of clarity, Definition 3.1 in [**Wol15**] carefully distinguishes the main classical versions – $\mathsf{TC1_n}$, $\mathsf{TC2_n}$, $\mathsf{TC3_n}$ – of Ravenel's conjecture (which is $\mathsf{TC1_n}$), and Theorem 3.2 shows the (well-known) implications between them. This was necessary, since when we localize these statements (Definitions 3.6 and 3.11) they are not all equivalent all the time, as shown in Theorem 3.12, until we again quantify over all n.

The punchline here (see Theorem B), is that all the versions of the localized telescope conjecture all hold, for all $n \geq 0$, in all the examples I investigated. By shifting between variations and using Theorem 3.12 (as well as Bousfield lattice computations), it was not that hard to prove these results. So in the end I was unable to find a counterexample to the local telescope conjecture(s). Proposition 3.16 shows that a counterexample to the local telescope conjecture(s) would imply a counterexample to the original telescope conjecture(s), though, so perhaps this isn't surprising!

Besides these topology-specific localized telescope conjectures, we also have the GSC and SDGSC. The GSC is a great question to ask in any tensor-triangulated category, and has fueled research in interesting non-topological categories. One of the key insights of [**Wol15**] is Theorem 3.5. I looked carefully at all the proofs that localization at a set of compact objects yields a smashing localization, and realized that by 'modernizing' the proofs I didn't need compactness, only strong dualizability. This gave birth to the SDGSC.

When the tensor unit is compact, strong dualizability and compactness are equivalent properties, so the SDGSC and GSC are the same thing. And many papers in tensor triangulated category theory are happy to assume the unit is compact. But localized categories of spectra, where the initial localization is not smashing, always yield tensor units that are not compact (and conversely, if the unit in the local category is not compact, then the localization is not smashing). So categories of local spectra are a perfect setting in which to tease apart the GSC and SDGSC. The punchline (see Theorem C) is that in several examples the GSC fails but the SDGSC holds.

While the SDGSC might be "the better thing" in this local setting, sadly it's necessary to point out that the SDGSC is also "not always the right thing". The only previous counterexample to the GSC, in Keller's [**Kel94**], is also a counterexample to the SDGSC. Keller looks at the derived category of a certain non-Noetherian ring, and in that setting the tensor unit is compact.

Finally, a few comments about the Bousfield lattice computations. They are directly relevant to the telescope conjecture, since $\mathsf{TC1_n}$ and $\mathsf{LTC1_n}$ are statements about Bousfield classes. Furthermore, since every smashing localization yields a pair of complemented classes, knowing something about the Bousfield lattice tells you something about the Boolean algebra of complemented classes, which tells you something about the smashing localizations.

1.3. The examples. The key insight in choosing examples was Corollary B.13 in [**HS99**], which says that there are no nonzero compact objects in the Z-local category for Z equal to any of the following: $\bigvee_{i\geq 0} K(i)$, $H\mathbb{F}_p$, I, BP or $H\mathbb{Z}$. These local categories are thus guaranteed to be well-generated but not compactly-generated. Furthermore, because there are no nonzero compact objects, any non-trivial smashing localizations on any of these local categories would imply failure of the GSC there. And this is exactly what happened (see Theorem C).

The case of $H\mathbb{Z}$ doesn't appear in the paper simply because I couldn't prove anything satisfying about it. I also include the cases of $K(n)$ and $E(n)$, reiterating the results of [**HS99**], mainly just for completeness. These two do provide compactly generated categories, in the first case because $L_{K(n)}F(n)$ is a compact generator and in the second because $L_{E(n)} = L_n$ is smashing.

Another reason for choosing these examples was that, except for BP, their Bousfield classes are "small" in the Bousfield lattice of $\mathcal{S}$. One of the key insights in relating Bousfield lattices of different categories is Lemma 2.8. It implies that if $\langle Z\rangle$ is small in $\mathsf{BL}(\mathcal{S})$, then $\mathsf{BL}(\mathcal{L}_Z)$ will be small. This is especially the case with $K(n)$, $H\mathbb{F}_p$, and I, and in fact this is how I prove that $\mathsf{BL}(\mathcal{L}_I)$ is the two-element lattice (Proposition 6.6).

In some ways, it seems like I chose the easy examples. But easy from a Bousfield lattice perspective certainly doesn't mean easy at the object level. One of the most fun results in the paper, Proposition 6.4, manages to take advantage of this discrepancy, and ties back to the hard question of classifying localizing subcategories.

References

[IK13] Srikanth B. Iyengar and Henning Krause, *The Bousfield lattice of a triangulated category and stratification*, Math. Z. **273** (2013), no. 3-4, 1215–1241.

[HS99] Mark Hovey and Neil P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100.

[Kel94] Bernhard Keller, *A remark on the generalized smashing conjecture*, Manuscripta Math. **84** (1994), no. 2, 193–198.

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414.

[Wol15] F. Luke Wolcott, *Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra*, Pacific J. Math. **276** (2015), no. 2, 483–509.

Department of Mathematics, Lawrence University, Appleton, WI 54915

E-mail address: luke.wolcott@lawrence.edu