Enchiridion: Mathematical User's Guides
`mathusersguides.com`
Volume 2 (September 2016)

A user's guide: Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra

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2. Metaphors and imagery

In [**Wol15b**] I also described some of the ways that I think about Bousfield lattices, tensor-triangulated categories, and localization. What I say here is a supplement not a repetition.

2.1. Localization and local categories. Algebraic topology is algebraic and topological because we use functors between algebraic and topological categories. One of the most basic is homology. Given a spectrum Z in the stable homotopy category of spectra $\mathcal{S}$, the Z-homology is a functor Z_* from $\mathcal{S}$ to graded abelian groups. We translate topological questions into algebraic questions, and often the algebra is easier to work with. Working in the world of graded abelian groups, we're using Z as a tool to understand $\mathcal{S}$. The Z-local category of spectra $\mathcal{L}_Z$ is what we get if we "pull back" from the world of graded abelian groups to topology. But through this process we've lost information, or we've simplified $\mathcal{S}$, depending on how you look at it. Two spectra X and Y are identified as "the same" in $\mathcal{L}_Z$ if they have the same Z-homology. A map between spectra becomes an equivalence in $\mathcal{L}_Z$ if and only if it is an isomorphism in Z-homology. In this sense, $\mathcal{L}_Z$ is "what $\mathcal{S}$ looks like, according to Z".

The paper [**Wol15a**] can be confusing, because we are looking at localizations of localizations. For example, what are the smashing localizations in the harmonic localization of $\mathcal{S}$? (This is answered in Theorem 4.4.) I found it most useful to really inhabit a/the local category $\mathcal{L}_Z$ as much as possible. We can take the local category as the "base" category, somewhat forgetting that it came from $\mathcal{S}$. The axiomatic approach of tensor triangulated category theory – and in general any work in category theory – slowly develops an ability to shift contexts, to inhabit and relocate and re-inhabit different categories without losing your luggage or sense of direction. I think this is one of the joys of category theory papers and results.

What is interesting is to see how intuition seeps between categories. Understanding a category seems to involve knowing various structures and properties it holds, as well as a slew of random results about it, and proof and computational techniques for it. Shifting between categories, say from $\mathcal{S}$ to $\mathcal{L}_Z$, or from $\mathcal{S}$ to the derived category $D(R)$ of a ring R, one loses many but not all of these. At the beginning of working on [**Wol15a**], I felt that $\mathcal{S}$ had a lot of color, personality, and character, but a local category $\mathcal{L}_Z$ was uncomfortably unknown and vague. After all, taking $Z = S^0$ gives $\mathcal{L}_Z = \mathcal{S}$, and taking $Z = 0$ gives $\mathcal{L}_Z = 0$, so how much can one say? Maintaining this ambiguity, I was able to build understanding of $\mathcal{L}_Z$, but only formally.

For example, Lemma 2.10, about computations in $\mathcal{S}$, makes sense to me and evokes various imagery and perhaps even feelings. The objects $F(n)$, $T(n)$, and $K(n)$ have their own personalities and are like characters in a story. But Lemma 3.7, which proves the same computations in $\mathcal{L}_Z$, is completely symbolic and formal for me. It makes sense. But because Z can be anything from 0 to S^0, I can't really get comfortable with these objects $LF(n)$, $LT(n)$, and $LK(n)$, although I can compute with them.

This captures well the experience of doing category theory. It is like studying "language", and choosing a handful of specific languages to focus on as examples. You understand each language to a different degree. The ones that you understand well evoke rich memories and associations, and speaking them is natural and sensuous. When you understand a language poorly, the understanding is formal, discontinuous, more rational and less unconscious. And now imagine taking up a new language, which is somehow close or related to one that you know well – some intuition carries over to the new setting. But that only gets you so far, and can certainly be misleading.

Only towards the end of working on [**Wol15a**], when I was choosing specific Z, did I feel like the personalities of the $\mathcal{L}_Z$ were coming out. The $K(n)$-local category is boring from a tensor-triangulated perspective. In terms of vague intuition and a category theory perspective, the $E(n)$-local category is like $n+1$ copies of the $K(n)$-local category. The harmonic category is like the $E(n)$-local category, where n has been sent off to infinity. Strange things start to happen in the harmonic category though, making it much more interesting than the $E(n)$ case – e.g. localization is not smashing, and the localizing subcategories get out of hand. The $H\mathbb{F}_p$-local category is probably my favorite in [**Wol15a**], since $H\mathbb{F}_p$ is like the weird cousin who writes fantasy novels while his normal cousins the $K(n)$ are watching television. If I could live in a local category, it would be the $H\mathbb{F}_p$-local one. Or maybe the I-local category, which is also bizarre, but in a more pathological way, and perhaps a little too mysterious and hard to understand.

2.2. Telescope conjectures and the Bousfield lattice. A lattice, vaguely speaking, is a nice partially ordered set. The Bousfield lattice is a lattice extracted from any well-generated tensor-triangulated category: each object X

gives a Bousfield class $\langle X \rangle = \{W \mid X \wedge W = 0\}$, and these form a lattice. In the context of [**Wol15a**], I found it useful to keep in mind the relationship between the Bousfield lattice and homological localizations. Given two spectra X and Y in $\mathcal{S}$, we have $\langle X \rangle \leq \langle Y \rangle$ in $\mathsf{BL}(\mathcal{S})$ if and only if L_X-locals are L_Y-locals. Of course this is the case if and only if L_Y-acyclics are L_X-acyclics (i.e. if $W \wedge Y = 0$ implies $W \wedge X = 0$), but for intuition I find the first characterization to be the better way to think about it. For computations, the second characterization is usually the more useful, but not always.

By definition, $\langle X \rangle = \langle Y \rangle$ if and only if L_X and L_Y are the same functor. The Bousfield lattice of a category T is a description of the different homological localizations that one can do on T. And this 'description' happens to have a lot of structure – e.g. it is a complete lattice, with additional operations like $- \wedge -$ and complementation.

I think of the telescope conjecture and all its variations, as described in Topic 1, as a wonderfully confusing mathematical story with lots of history. The original conjecture is hard and I have no hope of solving it. Now there are multiple versions, in different contexts. Many different people have very different perspectives and intuition about it. In my opinion, no one can really be said to *understand* the telescope conjecture, only different aspects of it.

The GSC and SDGSC are a little easier to grasp than the original topological versions $\mathsf{TC1_n}$, $\mathsf{TC2_n}$, and $\mathsf{TC3_n}$. They have a much more categorical feel. When I think of the GSC, I see a Venn diagram of a circle in a square. Every set of compact objects yields a smashing localization. It's like "every A is a B". The GSC asks if every smashing localization comes from a set of compact objects – "is every B an A?" What the SDGSC does is shift A to a slightly different A'. Theorem 3.5 says that every A' is a B, and the SDGSC asks if every B is an A'. But the relationship between A and A' – between compactness and strong dualizability – is category-dependent (see for example [**HPS97**, Theorem 2.1.3]). This is not a tightening of the GSC, but I guess I would call it an improvement, except for all those settings where the tensor unit is compact, in which case the GSC and SDGSC are equivalent, and A and A' are the same.

References

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610.

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