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A user's guide: Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra

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3. Story of the development

Careful background reading set the foundation. After reading [**IK13**], I became excited to look around for 'new' well generated tensor triangulated categories, in order to calculate their Bousfield lattices. Another result in [**IK13**], Proposition 2.1, points out that every Bousfield class $\langle Z \rangle$ of a well generated T is a well generated localizing subcategory. This implies that the Verdier quotient $\mathsf{T}/\langle Z \rangle$, which is equivalent to the local category $\mathcal{L}_Z$, is well generated. I found out about this early in fall 2012, and so began thinking about localized categories and their Bousfield lattices. I was in the mood to get back to topology, after writing a thesis on mostly homological algebra, so I started reading about localized categories of spectra.

I began this background reading in the first months of my postdoc position in fall 2012. Having earned my PhD in June in Seattle, and spent the summer traveling, I started a postdoc with Dan Christensen at the University of Western Ontario. However, Dan had a sabbatical year and so we spent the year as visitors at the Instituto Superior Técnico in Lisbon, Portugal. From September to December I worked through many different papers, planting seeds and looking for inspiration and good questions. A visit to the University of Bielefeld, to talk with Greg Stevenson about the harmonic category and with Henning Krause about the generalized smashing conjecture, was an important part of this. The cafés in Lisbon provided a great atmosphere for research, because not understanding Portuguese I was free from the distractions of written signs and overheard conversations. Moving alone to a new city in a new country provided lots of inspiration and free time.

Hovey and Strickland's paper [**HS99**] already answered all the questions I was interested in for the $K(n)$-local and $E(n)$-local categories. But studying their proofs gave me some ideas for other categories. Earlier in 2012, Jon Beardsley showed me that he had calculated the Bousfield lattice of the harmonic category.

I spent some time trying to classify the localizing subcategories of the harmonic category, but gave up.

During this time I was also reading about the "telescope conjecture" – really the GSC – in various algebraic categories. I started to become curious about the whole telescope conjecture story.

In January and February 2013 I was distracted working on other research projects, but this allowed me to gain some perspective. In March and April 2013, while spending weekends riding my motorcycle to the Portuguese beaches and cliffs to surf or rock climb, I dove into the history of all (or most of) the variations of the telescope conjecture. By early April I had figured out the right questions to ask, such as: what are the different versions of the TC, in local categories of spectra? What are the implications between them? What are the Bousfield lattices of these categories? They seem obvious to me in hindsight, but formulating these questions – realizing that they are interesting and probably answerable – was more than half the total work. It became clear that I most likely wouldn't be able to answer Ravenel's original conjecture $\mathsf{TC1_n}$ in $\mathcal{S}$. It also became clear that the variations had non-trivial implications between them, which might create subtleties in the local setting. Around this time, and heading into May, I formulated the different variations $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, and $\mathsf{LTC3_n}$ for localized categories of spectra. But it wasn't until later in the summer, in southern France, that I dug the proofs in Theorem 3.2 out of the literature, and tweaked them to give the implications in Theorem 3.12 and 3.13.

Starting in late April, I headed off from Lisbon on a Europe motorcycle trip. I spent a week at the University of Seville talking with Fernando Muro. I rode up to Madrid and flew to the University of Copenhagen for ten days, then flew back and continued north to Barcelona. I visited Carles Casacuberta there for a week in late May.

At each university I would give a seminar or conference talk, and I regularly video conferenced with my postdoc supervisor in Lisbon. Throughout the trip I was working in offices or cafés on my research. In between each mathematical visit, I spent one or two weeks rock climbing and camping – at El Chorro, Rodellar, and Siurana. From Barcelona I went to southern France, where I climbed in the Gorges du Verdon for ten days. This naturally inspiring, simple lifestyle was very conducive to immersive thought. It was during this time, in mid-June, that I finalized most of the results and wrote most of the paper, sitting at a booth in the village bar, after mornings spent climbing.

An exciting turning point occurred in mid-May in Spain. I was reading through all the different proofs from the last three decades that localization at a set of compact objects yields a smashing localization. By piecing together different proofs, and using the new results in [**IK13**], I realized that I didn't need compactness and could prove it with a set of strongly dualizable objects. This gave birth to the SDGSC. I already knew that the GSC failed in the harmonic category; I had shown this the previous fall. Now I had hope that maybe the SDGSC would hold where the GSC failed. Over the next few weeks I showed that this is the case in the examples I was considering.

Having made some progress, in early June I pulled together a list titled "big results I hope for". This served as a roadmap.

As mentioned above, my muse was kind to me during ten days in southern France. I believe the (mathematical) catalyst was having the implications of Theorems 3.2, 3.12, and 3.13 worked out, and then turning to the range of examples to try to answer the localized telescope conjectures. I also began the process of writing up my results, and so I pulled together many classical computations into one spot, Lemma 2.10. With the clarity of Lemma 2.10, and its local version Lemma 3.7, and by moving between the versions – $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, $\mathsf{LTC3_n}$, GSC, SDGSC – and between all the different examples – harmonic, $H\mathbb{F}_p$-local, I-local, etc. – I was able to tease out all the results of Section 6 without getting caught up anywhere for too long.

Also in southern France, Proposition 3.8 and Lemmas 3.9 and 3.10 emerged concurrently and quickly. After months of having a vague sense that something was going on (e.g. that l_n^f might always be smashing, or that perhaps $l_n = LL_n$ in some cases), the statements and proofs of all three of these results came together in a day or two. One of the last things I proved – although perhaps one of the more conceptually important – was Proposition 3.16. This wasn't necessarily because it was hard. Maybe a part of me didn't want to know Proposition 3.16, because it says that it would be very unlikely to find a counterexample to one of the localized telescope conjectures.

After this binge of work, I took a week break to ride my motorcycle through the Italian and Swiss Alps. Unable to not think about the paper, though, I ended up continuing to write up the results, which led to thinking more about how to expand or improve what I had. After all, it wasn't clear to me where to stop. Most of the things on my list of "big results I hope for" were settled, but some refused to yield. After trying a bit more to no avail, I left some of them in the paper as Questions, and left some in my personal notebooks as future questions to pursue. From this point, all that remained was to write up the Preliminaries and Introduction, and submit. The final touches were done while attending the Young Topologist's Meeting in Lausanne in early July 2013. After riding up to Copenhagen to collaborate on a math-art project, and then to Berlin, I let the paper sit for a week or two and then submitted it from Berlin at the end of July.

Several months after submission, I returned to look at some of the unanswered questions. I had calculated the Bousfield lattice of the $H\mathbb{F}_p$-local category to be a two-element lattice, but had a feeling that the $H\mathbb{F}_p$-local category should have proper non-zero localizing subcategories. If I could find one, it would be a localizing subcategory that wasn't a Bousfield class, and this would be exciting and interesting. In December 2013 I posted a question to this effect on MathOverflow, and Mark Hovey quickly answered it, giving a proper non-zero localizing subcategory in the $H\mathbb{F}_p$-local category. This improvement (which I wasn't able to build on very much) became Proposition 6.4 in the final version of [**Wol15**].

References

[IK13] Srikanth B. Iyengar and Henning Krause, *The Bousfield lattice of a triangulated category and stratification*, Math. Z. **273** (2013), no. 3-4, 1215–1241.

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