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A user's guide: Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra

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4. Colloquial summary

A "category" is a collection of mathematical objects, with "morphisms" – functions, or relationships – between them. It's a way of describing different mathematical "worlds" in which to work. Topologists work in categories where the objects are different types of spaces. Stable homotopy theory is a type of topology, where we study the category of "spectra", denoted $\mathcal{S}$. A spectrum is a strange sort of generalization of what we normally think of as space (one that allows for "negative-dimensional space", but that's another story...). The paper under discussion, [**Wol15**], is a paper about stable homotopy theory, about spectra.

Rather than consider the usual category of spectra, I do something to it – I "localize" it. For each spectrum, call it Z, it is possible to localize $\mathcal{S}$ at Z, and we get the "Z-local category of spectra", denoted $\mathcal{L}_Z$. Intuitively, this category is "what $\mathcal{S}$ looks like to Z". The intuition behind localization agrees with the everyday notion: it's a way of simplifying $\mathcal{S}$, of focusing on particular aspects of it. In [**Wol15**], I consider a bunch of different local categories: the BP-local category, the $H\mathbb{F}_p$-local category, the I-local category, etc. Each of BP, $H\mathbb{F}_p$, and I are different spectra, and their local categories are quite different. Each has special properties and characteristics, and each can tell us something different about the larger, un-localized category of spectra $\mathcal{S}$.

So in the paper I'm traveling between local categories of spectra, but what particular questions am I asking in each local setting? There are two main ones, related to each other.

One is computational. Each of these local categories of spectra has a "Bousfield lattice" associated to it. I want to know what it is. It's like saying, "Every town in Fox county has a ZIP code. What are these ZIP codes?" In all the examples I consider in the paper, I'm able to calculate the Bousfield lattice completely (except in one case, where all I can do is say something about how big it is). This leads to some interesting consequences (for example, Proposition 6.4: "In the $H\mathbb{F}_p$-local category, there is a localizing subcategory that is not a Bousfield class").

The second type of question is more elaborate. In a famous paper in 1984, the stable homotopy theorist Doug Ravenel announced a list of conjectures about spectra – good, hard questions worth pursuing. In the 1980s and 1990s all of them were answered, by various people, except one: the "telescope conjecture".

Roughly speaking, what is the telescope conjecture? It's a question about "smashing localizations". A smashing localization is an especially nice type of localizing – it results in a local category that is particularly nice in relationship to the larger category. The telescope conjecture aims to classify these nice localizations.

Rather than tackle the original telescope conjecture, a question in $\mathcal{S}$, my paper translates this telescope conjecture into the local categories, and asks it there. Sure enough, the localized telescope conjecture is easier to answer. Much of my paper is devoted to this purpose. But things have gotten complicated since 1984. Instead of one telescope conjecture, there are now four – all slightly different versions of the original. And to make matters worse, I invent a new one.

I'm able to answer all five versions of the conjecture in all the local categories I consider. For four of the five conjectures (including the one I invented), the answer is: yes, the conjecture holds, in all the categories considered. And I use my Bousfield lattice calculations in key ways to do this. However, one of the conjectures (the "GSC") fails, in several of the local categories I consider, and this is also interesting.

References

[Wol15] F. Luke Wolcott, *Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra*, Pacific J. Math. **276** (2015), no. 2, 483–509.

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