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A user's guide: The Adams-Novikov E_2-term for Behrens' spectrum $Q(2)$ at the prime 3

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1. Key insights and central organizing principles

The goal of [**Lar15**] is to compute the E_2-term of a spectral sequence that converges to the homotopy groups of a spectrum called $Q(2)$. This E_2-term, which I denote $E_2^{*,*}Q(2)$, is tied to the stable homotopy groups of spheres π_*S. The latter are known more briefly as the stable stems. You could say there are "six degrees of separation" between $E_2^{*,*}Q(2)$ and the stable stems, because:

(1) $E_2^{*,*}Q(2)$ approximates $\pi_*Q(2)$ via the spectral sequence,
(2) $\pi_*Q(2)$ is the source of a homomorphism to $\pi_*L_{K(2)}Q(2)$ (where $L_{K(2)}$ is localization at the 2nd Morava K-theory at the prime 3),
(3) $\pi_*L_{K(2)}Q(2)$ is in a l.e.s. of homotopy groups with $\pi_*L_{K(2)}S$ arising from a cofiber sequence,
(4) $\pi_*L_{K(2)}S$ is the target of a homomorphism from π_*L_2S arising from a homotopy pullback square (where L_2 is localization at the 2nd Johnson-Wilson theory $E(2)$ at the prime 3),
(5) π_*L_2S is a second-order approximation to $\pi_*S\otimes\mathbb{Z}_3$ by the Chromatic Convergence Theorem of Hopkins and Ravenel, and

(6) $\pi_* S \otimes \mathbb{Z}_3$ approximates $\pi_* S$ just as looking at the 3-torsion of any group gives you some information about that group.

So, my computation is in the same mathematical neighborhood as $\pi_* S$, if only at the outskirts, and that's an exciting place to live! Even more exciting is the surprising role that number theory plays in this computation and in the study of the stable stems in general. One of the bridges connecting number theory and homotopy theory is a spectrum known as topological modular forms, which happens to be a key ingredient in the construction of $Q(2)$. As a result, the organizational structure of my computation is governed completely by elliptic curves, modular forms, and certain maneuvers one can make with them. One such maneuver underlying the construction of $Q(2)$ is a degree 2 isogeny between elliptic curves (i.e., a surjective morphism with a kernel of size 2), which is why the "2" appears in the notation for $Q(2)$. By contrast, the "2" in $K(2)$ and $E(2)$ refers to chromatic level, an idea to be explored later in this user's guide.

Although the notation does not indicate it, the spectrum $Q(2)$ is, for our purposes, linked specifically to the prime 3 as the above list indicates. However, it is one of an infinite family of "Q-spectra" $\{Q(\ell)\}$ obtained by letting the prime p and the isogeny degree ℓ vary. These spectra were constructed by Mark Behrens [**Beh06**] in order to shed light on the p-torsion of the stable stems at all different primes p. We shall see that different things are known about a given $Q(\ell)$ depending on the values of p and ℓ.

1.1. From the Steenrod algebra to BP-theory and elliptic curves. The starting point of my computation is a Hopf algebroid, a certain algebraic structure. The classical Adams spectral sequence starts with an object of a similar algebraic nature, so let's begin there.

Let p be a prime and let A_* denote the mod p dual Steenrod algebra. The pair $(\mathbb{F}_p, A_*)$ forms a Hopf algebra over $\mathbb{F}_p$. One interpretation of this Hopf algebra structure is to say that, for an $\mathbb{F}_p$-algebra T, the sets $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p, T)$ and $\mathrm{Hom}_{\mathbb{F}_p}(A_*, T)$ are the objects and morphisms, respectively, of a category. The well-known coproduct $\Delta : A_* \to A_* \otimes_{\mathbb{F}_p} A_*$ corresponds to composition of morphisms. Mind you, the category in this case is not exciting, for it only has one object! But it turns out that all morphisms are invertible. In other words, the category is really just a *group* with the obvious map $\eta : \mathbb{F}_p \to A_*$ (called the unit map) corresponding to the group's unit element. On the category level, η "picks out" from a given morphism the single object that acts as both its source and target. The Hopf algebra cohomology of $(\mathbb{F}_p, A_*)$ is precisely the cohomology of the cobar complex

$$\mathbb{F}_p \xrightarrow{\eta} A_* \to A_* \otimes A_* \to A_* \otimes A_* \otimes A_* \to \cdots$$

constituting the E_2-term for the Adams spectral sequence converging to $\pi_* S \otimes \mathbb{Z}_p$.

Passing from the classical Adams spectral sequence to a tool involving an exotic cohomology theory, such as the Adams-Novikov spectral sequence (ANSS) based on BP-theory [**Rav86**], means changing the input format from a Hopf algebra to something slightly more general. The pair of $\mathbb{Z}_{(p)}$-algebras (BP_*, BP_*BP) has the property that, given a $\mathbb{Z}_{(p)}$-algebra T, the sets $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(BP_*, T)$ and $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(BP_*BP, T)$ are the objects and morphisms of a category. All morphisms are once again invertible. This time, though, there is the potential for more than one object, which means the category is a *groupoid*. So the pair is called a *Hopf algebroid* rather than a Hopf algebra. Appropriate, no? Since in this case one can "pick out" either the source or target of a morphism as they are generally different, there are two distinct structure maps $\eta_R, \eta_L : BP_* \to BP_*BP$, the right and left units. The algebra BP_*BP is a left BP_*-module via η_L and a right BP_*-module via η_R. The Hopf algebroid cohomology of (BP_*, BP_*BP) is precisely the cohomology of the cobar complex

$$BP_* \xrightarrow{\eta_R - \eta_L} BP_*BP \to BP_*BP \otimes BP_*BP \to \cdots$$

that constitutes the E_2-term for the ANSS converging to $\pi_*S \otimes \mathbb{Z}_{(p)}$. For an arbitrary spectrum X in place of the sphere S, the ANSS input would be $BP_*(X)$ instead of $BP_*(S) \cong BP_*$, and the target would be $\pi_*X \otimes \mathbb{Z}_{(p)}$.

The spectrum of topological modular forms TMF is an object (depending on p) whose homotopy groups π_*TMF can be interpreted collectively as a refinement of modular forms over $\mathbb{Z}_{(p)}$, hence the name. We shall now see why this interpretation is reasonable.

For the remainder of this section let's fix $p = 3$, as that is the setting for my computation. The ANSS for $X = TMF$ rather than $X = S$ has as its E_2-term the cohomology of a Hopf algebroid over $\mathbb{Z}_{(3)}$ that I denote (B, Γ), where

$$\begin{aligned} B &= \mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}]/(\Delta = q_4^2(16q_2^2 - 64q_4)), \\ \Gamma &= B[r]/(r^3 + q_2r^2 + q_4r). \end{aligned}$$

For fixed values of q_2 and q_4 the equation $y^2 = 4x(x^2 + q_2x + q_4)$ is a Weierstrass equation for a non-singular elliptic curve, and for a fixed r the map $x \mapsto x + r$ is an elliptic curve isomorphism that preserves this particular Weierstrass form [**Sil09**]. Therefore, the groupoid obtained by mapping into a $\mathbb{Z}_{(3)}$-algebra T has certain elliptic curves over T as its objects, and isomorphisms $x \mapsto x + r$ as its morphisms.

Modular forms enter the picture because B itself *is* the ring of modular forms over $\mathbb{Z}_{(3)}$ with respect to the congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}(2, \mathbb{Z})$ [**DS05**]. Moreover, the ANSS E_2-term for TMF is the cohomology of the cobar complex

$$B \xrightarrow{\eta_R - \eta_L} \Gamma \to \Gamma \otimes \Gamma \to \ \Gamma \otimes \Gamma \otimes \Gamma \to \cdots \tag{1}$$

and a classical computation shows that the 0th cohomology (i.e., $\ker(\eta_R - \eta_L)$) has the form

$$MF = \mathbb{Z}_{(3)}[c_4, c_6, \Delta, \Delta^{-1}]/(1728\Delta = c_4^3 - c_6^2)$$

which is precisely the ring of modular forms over $\mathbb{Z}_{(3)}$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. By the time one arrives at π_*TMF one sees many of these modular forms in one guise or another (e.g., there is a homotopy class corresponding to a multiple of Δ) plus other stuff. The complete computation of $\pi_* TMF$ is due originally to Hopkins and Miller and is beautifully described in expository works by Tilman Bauer [**Bau08**], Akhil Mathew [**Mat**], and Andre Henriques [**Hen**].

The spectrum TMF has several close cousins, one of which has as its underlying Hopf algebroid the pair (B, B) where $\eta_R = \eta_L = 1$ (an example of a "trivial" Hopf algebroid). In this case each elliptic curve C from the groupoid comes with the additional datum of a choice of order 2 subgroup H, and no morphism $x \mapsto x + r$ can preserve this choice. The ANSS collapses at E_2 and the homotopy groups are described by the graded algebra B itself (the grading of an element determines its degree in homotopy). This spectrum is denoted $TMF_0(2)$.

KEY IDEA 1.1. *The spectra TMF and $TMF_0(2)$ are E_∞-ring spectra, so they receive a map from S and are therefore tied to $\pi_* S$ via the induced homomorphism on homotopy. Moreover, their homotopy groups are fully computable precisely because the algebras constituting their underlying Hopf algebroids are finitely generated, as opposed to $(\mathbb{F}_p, A_*)$ and (BP_*, BP_*BP) whose full cohomologies (let alone $\pi_* S \otimes \mathbb{Z}_p$) will be forever out of reach.*

1.2. Constructing $Q(2)$. To tie the construction of $Q(2)$ to something familiar, imagine an ordered 2-simplex with vertices $V = \{v_0, v_1, v_2\}$ as shown in Figure 1. Let E denote the set of its edges and F denote the singleton set consisting of its one and only two-dimensional face f. [For technical reasons, E and F actually contain additional phantom elements that are safely ignored—see the second paragraph after this one.] There are two collections of "face" maps

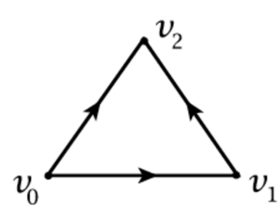

FIGURE 1. An ordered 2-simplex.

attached to this structure: the first is a triple $d_0, d_1, d_2 : F \to E$ where $d_i(f)$ is the edge opposite v_i; the second is a pair $d_0, d_1 : E \to V$ (a blatant abuse of notation) where d_i sends an edge to the vertex opposite its i-th one. Face maps can be composed and they satisfy identities that are straightforward but tedious to write down. We can sum this up with a diagram

$$V \leftleftarrows E \Lleftarrow F \tag{2}$$

displaying the two collections of face maps.

Now imagine applying the free abelian group functor Free$(-)$ to (2), in which case one might denote Free(d_i) by δ_i. Because the maps δ_i are group homomorphisms, they can be added or subtracted as long as they share the same source and target. For example, taking a cue from basic homology, one might take alternating sums of the δ_i, yielding

$$\mathrm{Free}(V) \Leftarrow \mathrm{Free}(E) \Lleftarrow \mathrm{Free}(F) \tag{3}$$

where the doubly thick arrow $\Leftarrow$ represents $\delta_0 - \delta_1$ and the triply thick arrow $\Lleftarrow$ represents $\delta^0 - \delta^1 + \delta^2$.

Diagrams (2) and (3) are each examples of a *simplicial set* (the latter having the additional structure of a simplicial *free abelian group*) that I hope demonstrate that such objects are natural. They happen to contain interesting stuff only in dimensions 0, 1, and 2. In general, simplicial sets can have interesting higher-dimensional stuff, and they also (always) have "degeneracy" maps that go in the direction opposite to the face maps. One of the degeneracy maps in the 2-simplex would send the vertex v_0 to the "degenerate" edge between v_0 and itself, for example. [This degenerate edge at v_0 is one of the aforementioned phantom elements.] However, this aspect of a simplicial structure is not relevant to my computation.

The construction of $Q(2)$ starts with a simplicial *stack*

$$\mathcal{M}_0 \Leftarrow \mathcal{M}_1 \Lleftarrow \mathcal{M}_2 \tag{4}$$

whose structure mirrors (2), but with sets replaced by stacks, hence its name. [A *moduli stack*, or sometimes just *stack*, is an algebraic construction useful for studying parametrized families of mathematical objects.] The $\mathcal{M}_i$ are moduli stacks of elliptic curves and the face maps between them are morphisms of stacks (see below for more on the genesis of these morphisms). The doubly- and triply-thick arrows once again denote alternating sums of the face maps. The precise name for (4) is "semi-simplicial stack," because there are no degeneracy maps in the data, and that is what the prefix "semi-" indicates in this context.

The semi-simplicial stack (4) can be realized topologically. This means there is a functor (contravariant, so arrows get turned around) that takes the diagram of stacks above, to a diagram of spectra. The result is a semi-cosimplicial spectrum

$$TMF \Rightarrow TMF_0(2) \vee TMF \Rrightarrow TMF_0(2). \tag{5}$$

made from the spectra TMF and $TMF_0(2)$ introduced in the previous subsection. This time, $\Rightarrow$ and $\Rrightarrow$ denote alternating sums of coface maps, and the prefix "semi-" indicates that there are no codegeneracy maps in the data.

The spectrum $Q(2)$ is the homotopy limit of the diagram (5).

1.3. The ANSS for $Q(2)$. Recall that the ANSS inputs for TMF and $TMF_0(2)$ are the cohomologies of their respective Hopf algebroids (B, Γ) and

(B, B). The key organizing principle of my paper is the ability to pass covariantly from (5) (or contravariantly from (4)) to a diagram of underlying Hopf algebroids

$$(B, \Gamma) \Rightarrow (B, B) \oplus (B, \Gamma) \Rrightarrow (B, B) \tag{6}$$

where $\Rightarrow$ and $\Rrightarrow$ are alternating sums of Hopf algebroid maps.

Key Idea 1.2. *Each Hopf algebroid map in* (6) *encodes a certain maneuver one can make with elliptic curves and their Weierstrass equations, and is algebraically straightforward.*

For example, one of the two maps $TMF \to TMF \vee TMF_0(2)$ is built from a map $TMF \to TMF_0(2)$ which is underlain by a map of Hopf algebroids

$$\phi_f : (B, \Gamma) \to (B, B)$$

that corresponds to "forgetting" the choice of order 2 subgroup within an elliptic curve. The map ϕ_f is, in turn, induced by the algebra map $\Gamma \to B$ that is the identity on B and sends r to 0.

Given (5) and (6), it is plausible that $E_2^{*,*}Q(2)$ can be assembled from the ANSS E_2-terms for TMF and $TMF_0(2)$, and that is indeed the case. That allows me to make my way from (6) to $E_2^{*,*}Q(2)$ by first replacing each Hopf algebroid in (6) by its cobar complex and multiplying the cobar complex differentials d for (B, Γ) by -1 in the middle column. This yields a double complex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle 0\oplus(-d)} & & \uparrow{\scriptstyle 0} \\
\Gamma \otimes \Gamma & \longrightarrow & 0 \oplus \Gamma \otimes \Gamma & \longrightarrow & 0 \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle 0\oplus(-d)} & & \uparrow{\scriptstyle 0} \\
\Gamma & \longrightarrow & 0 \oplus \Gamma & \longrightarrow & 0 \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle 0\oplus(-d)} & & \uparrow{\scriptstyle 0} \\
B & \longrightarrow & B \oplus B & \longrightarrow & B
\end{array} \tag{7}$$

where the horizontal maps are induced by the corresponding alternating sums of Hopf algebroid maps in (6). The E_2-term $E_2^{*,*}Q(2)$ is the cohomology of the totalization of (7).

1.4. Computing the ANSS for $Q(2)$. The construction from the previous subsection implies that $E_2^{*,*}Q(2)$ is computable via the double complex spectral sequence (DCSS) for (7). The first step in this DCSS is to take cohomology with respect to the vertical arrows. If Ext^n denotes the n-th cohomology of (B, Γ) (i.e., the n-th cohomology of the cobar complex (1)) by (which, by homological algebra, is a sensible abbreviation) then, since $\mathrm{Ext}^0 \cong MF$, this yields

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Ext}^2 & \longrightarrow & \mathrm{Ext}^2 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Ext}^1 & \longrightarrow & \mathrm{Ext}^1 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ MF & \longrightarrow & B \oplus MF & \longrightarrow & B \end{array} \tag{8}$$

The next step in the DCSS is to take cohomology with respect to the horizontal arrows in (8). Fortunately, much of this work is trivial.

KEY IDEA 1.3. *The groups* Ext^n *are entirely 3-torsion for* $n \geq 1$*, and the horizontal arrows in* (8) *are induced by Hopf algebroid maps that send polynomial algebra generators to multiples of 3, hence identically zero. Therefore, the only non-trivial computation in this step of the DCSS is the cohomology of the 0th row*

$$MF \to B \oplus MF \to B. \tag{9}$$

Because there are only three columns of data in these diagrams, the DCSS has only one additional step past measuring cohomology horizontally, after which the DCSS stabilizes. This additional step requires that I compute the kernel and the image of just a single map (i.e., DCSS differential) whose source is Ext^1 and whose target is the cokernel of the map $B \oplus MF \to B$ from (9).

With a few isolated exceptions, I completely compute the DCSS and obtain a presentation for $E_2^{*,*}Q(2)$ in my paper. The answer is a jungle of 3-torsion that I will not reproduce here (see the main theorem of [**Lar15**]).

1.5. Why $Q(2)$? One key property of $Q(2)$ is the existence of a cofiber sequence

$$L_{K(2)}DQ(2) \to L_{K(2)}S \to L_{K(2)}Q(2) \tag{10}$$

with the $K(2)$-localization of $Q(2)$ on the far right and the $K(2)$-localization of the Spanier-Whitehead dual of $Q(2)$ on the far left. This is a big reason why $\pi_*Q(2)$ is worth pursuing. For other values of p and ℓ, analogous cofiber sequences involving $Q(\ell)$ are only conjectured to exist.

The cofiber sequence (10) is not only a reason to study $Q(2)$, it is also a reason why $Q(2)$ was built in the first place. It turns out that (10) (along with its conjectured analogs for other p and ℓ) is inspired by a pair of earlier results.

The first is a result of Adams/Baird and Ravenel that gives, at the prime 2 for example, a cofiber sequence

$$L_{K(1)}S \to KO_2 \to KO_2 \tag{11}$$

(here KO_2 is 2-adic real K-theory). The second is a result of Goerss, Henn, Mahowald, and Rezk, at the prime 3 [**GHMR05**]; namely, there exists a diagram

$$L_{K(2)}S \to X_1 \to X_2 \to X_3 \to X_4 \to X_5 \tag{12}$$

that induces a tower of cofiber sequences with $L_{K(2)}S$ at the top. Behrens' construction of $Q(2)$ as a semi-cosimplicial object is a reinterpretation of the machinery underlying (12) that is more in the spirit of (11) and more explicitly number-theoretic. For example, TMF appears in Goerss-Henn-Mahowald-Rezk but is not identified outright as a number-theoretic object. It appears there in the guise of a certain homotopy fixed point spectrum.

2. Metaphors and imagery

In the mathematical neighborhood of π_*S, I like to imagine the elements of the stable stems all living in a tall building at the center of town. Each lives happily in its own apartment. There must be infinitely many units, then, so let's imagine infinitely many floors (I said tall, right?) and an infinitely long hallway on each floor. In particular, just like Hilbert's eponymous hotel, this building always has vacancies! From that you might surmise it's a warm and welcoming place, where you can walk in and meet some of the tenants, get to know them, find out what they're about, and maybe make some friends. You would not be totally wrong. Walk into the lobby on floor 0 and you find everyone is genuine and forthcoming and glad to tell you about themselves. Someone invites you in for a delicious dinner. Mmmmm. But higher up, things start to change. Folks on floor 1 invite you in for wine and cheese, which is great, but they all have dinner plans with some guy named Adams so you have to leave after the cheese. When you're seen roaming the hall on floor 2 some tenants kindly offer you directions because you look lost. Directions...to the exit! Burn! No cheese for you, unless you happen to be a high-powered homotopy theorist who used to hang out at Princeton in the 1970s. On floor 3 the air is stuffier. The residents pay you no mind because they are kind of a big deal; after all, several of them were featured in a New York Times article back in 1976 [**NYT**]. Undaunted, but perhaps a little hungry, you press on to floor 4. The elevator door opens and a huge bouncer blocks your way. You crane your neck hoping merely for a glimpse of the hallway, but no dice. You ask who lives on this floor: he says nobody. You ask how to get to the higher floors: he says there are no higher floors, which is a lie. He then says "you were never here" and next thing you know you wake up back in the lobby, dazed and confused.

That escalated quickly. Floor 3 is indeed exclusive, and floors ≥ 4 are basically secret societies. Do not despair, however. It's not you. For all practical purposes, *nobody* from the outside has ever been allowed on the upper floors, not even the distinguished topologists who commissioned the building's construction in the first place. What happens up there is in large part a mystery.

2.1. Wine? Cheese? In this apartment building analogy, the tenants (a.k.a. elements of π_*S) on floor n are those detected by $K(n)$-local sphere but not by the $K(n-1)$-local sphere, across all primes p. It is morally true, but not quite precise, to say these are exactly the elements of π_*S represented by cohomology classes on the n-line of the ANSS E_2-page for the sphere across all primes p. And as algebraic topologists push their way forward through each $K(n)$-localization, or climb their way up the ladder of the ANSS, computational efforts work fine at first but then promptly grind to a halt at about 4 steps in. As a result, most concrete computations (like the one in my paper) take place on a lower floor.

Key Idea 2.1. *The ANSS for the sphere and the Morava K-theories $K(n)$ (and therefore the organizational structure of our apartment analogy) are underlain by the theory of formal group laws* [**Haz12**]. *The link between formal group laws and stable homotopy originated in the work of Novikov and Quillen* [**Qui69**]. *The increase in computational complexity as n increases can be tied to an increasingly difficult sequence of group cohomology computations, where the groups in question are automorphism groups of certain formal group laws of increasing complexity (i.e., height—see Definition A2.2.7 in* [**Rav86**]*).*

In my paper, I wander the 2nd floor of this apartment building hoping to learn more about its 3-torsion residents. Amazingly, most of the 2nd floor inhabitants are related to one another—that is, they are a *family*—that homotopy theorists call the beta family. There is a conjectured link between the 3-torsion members of this family and modular forms with certain properties. We are hopeful that our computation may help elucidate this link in future work (at primes ≥ 5 this link is known to exist). Even more amazingly, there are analogous families of elements on floor 1 and on the higher floors with either proven or conjectured links to certain number-theoretic objects. These ideas lie on the cutting edge of homotopy theory. We'll take a tour of the entire building (to the extent we can) from the bottom up, with an eye toward this cutting-edge technology. This should help put my paper in its proper context.

2.2. The lobby. The zeroth Morava K-theory $K(0)$ is equivalent to rational homology $H\mathbb{Q}$. This means $L_{K(0)}S$ detects non-torsion, i.e., it detects $\pi_0 S \cong \mathbb{Z}$. Therefore the lobby-level residents are the integers! No wonder they're so friendly and inviting.

2.3. The first floor. In [**Ada66**] Adams used K-theory to compute the image of the stable J-homorphism

$$J : \pi_*(SO) \to \pi_* S$$

from the homotopy of the stable orthogonal group to the stable stems. The elements of the image of J all live on floor 1 and are generated by a family of related elements known as the alpha family, the first and most thoroughly understood of the *Greek letter families*. The $K(1)$-local sphere ($K(1)$ itself being a summand of K-theory completed at p) zooms in on the alpha family elements. We will linger on floor 1 for a bit to describe how the alpha family is organized, since the higher Greek letter families have similar organizational structures. Might as well enjoy some wine and cheese, too.

It is easiest to first fix a prime p and discuss the p-torsion of the alpha family, so let's do it. Then in the ANSS there is a collection of elements $\{\alpha_i^{alg}\}$ indexed by integers $i \geq 1$, where

$$\alpha_i^{alg} \in \mathrm{Ext}^{1,2pi-2i}_{BP_*BP}(BP_*, BP_*)$$

so these elements live on the 1-line. In the chromatic spectral sequence (CSS), which is an SS converging to the ANSS E_2-page that we will treat like a black box in this discussion, α_i^{alg} is born out of a "fraction" of the form

$$\frac{v_1^i}{p}$$

and so the subscript i determines the power on v_1. Because you can go from one member of the collection to the next by multiplying by v_1 in the CSS, this collection is called a v_1-periodic family (think of Bott periodicity). They are all non-trivial elements of the ANSS E_2-page; in fact, they all have order p.

The ANSS is such an algebraic jungle, that the existence of this infinite yet intimately connected network of nontrivial elements is miraculous. But it gets better. It turns out each α_i^{alg} survives the ANSS and yields a homotopy element $\alpha_i \in \pi_{2pi-2i-1}S$. So the *alpha family at* p, i.e., the collection $\{\alpha_i^{alg}\}$, yields a corresponding collection $\{\alpha_i\} \subset \pi_*S \otimes \mathbb{Z}_{(p)}$. Taking the union of these latter sets over all primes yields the family of wine and cheese enthusiasts that occupies most of floor 1 of the building.

Recall that if

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

then B_n is the n-th *Bernoulli number*. The sequence $\{B_n\}$ is significant in number theory, from summing up the m-th powers of the first k integers to Fermat's Last Theorem, and many things in between. There turns out to be a close relationship between the alpha family at odd primes (the situation at $p = 2$ is muddier) and Bernoulli numbers.

Key Idea 2.2. *Let p be any prime.*

(1) *Each member of the alpha family at p yields a non-trivial element of π_*S of order p, and they collectively generate the image of J at p (although if $p = 2$, neither of these assertions is 100% true).*
(2) *The alpha family element α_i^{alg} is divisible by $j - 1$ powers of p, where j is 1 plus the number of powers of p dividing i.*

(3) *If p is odd, there is a correspondence between the alpha family at p and Bernoulli numbers, in the sense that the order p^j of α_i^{alg}/p^{j-1} is the p-factor of the denominator of B_t/t where $t = pi - i$ and B_t is the t-th Bernoulli number.*

2.4. The second floor. For a fixed p, the *beta family at p* $\{\beta_i^{alg}\}$ lives on the 2-line of the ANSS. The provenance of the beta family is analogous to what occurred one floor below with the alpha family, since β_i^{alg} comes from a "fraction"

$$\frac{v_2^i}{v_1 \cdot p}$$

in the CSS. The beta family is therefore a v_2-periodic family.

Ideally, the beta family $\{\beta_i^{alg}\}$ across all primes would consist entirely of non-trivial elements that survive the ANSS and yield non-trivial homotopy elements $\{\beta_i\} \subset \pi_* S$. But these are the same $\beta_i s$ that couldn't wait to boot you out the door! We know they're a tad dodgy, and therefore unlikely to exhibit behavior as consistent as the alpha family. This intuition is sound. In fact, at the very first prime ($p = 2$), the very first beta element

$$\beta_1^{alg} \in \mathrm{Ext}^{2,4}_{BP_*BP}(BP_*, BP_*)$$

is zero! To guarantee all beta elements in the ANSS are non-trivial, p must be at least 3, and to guarantee they all yield non-trivial homotopy elements p must be at least 5.

Recall that a *modular form over $\mathbb{Z}$* is a function $f : \mathfrak{h} \to \mathbb{C}$ on the upper half plane $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ satisfying

$$f(\gamma z) = (cz + d)^k f(z)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, as well as a growth condition at $i\infty$. The "over $\mathbb{Z}$" part means that the Fourier expansion obtained from the nice periodicity property $f(z + 1) = f(z)$, namely

$$f(q) = \sum_{i=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$, has integer coefficients. Modular forms must satisfy so many symmetries simultaneously that their existence is miraculous, but they're out there. And, like Bernoulli numbers, the Fourier coefficients of modular forms encode a lot of number-theoretic information. It turns out that modular forms are to the beta family what Bernoulli numbers are to the alpha family.

Key Idea 2.3. (1) *If $p \geq 5$, each member of the beta family at p yields a non-trivial element in $\pi_* S$ of order p.*

(2) *Certain beta elements β_i^{alg} have representatives in the CSS that can be further divided by v_1 and p, and if $p \geq 5$, there is a 1-1 correspondence*

between these "divided" beta family elements and modular forms over $\mathbb{Z}$ up to certain congruences depending on p.

(3) *In my paper I give evidence that $\pi_* Q(2)$ detects "divided" beta family elements at the prime 3 by finding candidate detecting elements on the level of Adams-Novikov E_2-terms. My hope is that this will eventually lead to a 3-primary version of the aforementioned 1-1 correspondence.*

The correspondence with modular forms is due to Behrens [**Beh09**], and is one of my main motivations for studying $Q(2)$ at the prime 3. Moreover, in the course of proving this 1-1 correspondence Behrens shows that the divided alpha and beta families at $p \geq 5$ are detected by the $E(2)$-localized unit map

$$\pi_* L_{E(2)} S \to \pi_* Q(\ell)$$

for appropriate values of ℓ depending on p. We would like to know whether something analogous is true at $p = 2$ and $p = 3$.

As I mentioned above, modular forms actually exist, and now is a great time to exhibit one. For a positive integer t consider the q-expansion

$$E_t(q) = 1 - \frac{2t}{B_t} \sum_{n=1}^{\infty} \sigma_{t-1}(n) q^n$$

where $\sigma_{t-1}(n)$ is the sum of the $(t-1)$-st powers of the divisors of n. If $t \geq 4$ then $E_t(q)$ is in fact a modular form of weight t called the *Eisenstein series of weight t*. If $p \geq 5$ then the Eisenstein series E_{p-1} (called the *Hasse invariant*) has the property that $E_{p-1} \cong 1 \mod p$, which means multiplication by E_{p-1} (modular forms have a ring structure—they can be added and multiplied) takes you from weight k modular forms to weight $k + p - 1$ modular forms without changing the q-expansion modulo p. This of course produces *congruences* between the q-expansions of modular forms, and turns out to be a key ingredient of Behrens' proof in [**Beh09**]. Unfortunately the q-expansion E_{p-1} doesn't quite accomplish this if p is 2 or 3. For example, while E_2 can be regarded as a modular form in a certain sense (different from the sense discussed here) it does not have the requisite properties to help produce congruences. The hope is that $Q(2)$ can come to the rescue, at least the prime 3.

2.5. The third floor and higher. The pattern of the Greek letter family construction might now be clear. The gamma family $\{\gamma_i^{alg}\}$ at p lives on the 3-line of the ANSS and is a v_3-periodic family, as each γ_i^{alg} originates as a fractional element

$$\frac{v_3^i}{v_2 \cdot v_1 \cdot p}$$

in the CSS. The somewhat-unpredictable behavior exhibited by the beta family continues here; for example, at the prime 2 the element

$$\gamma_1^{alg} \in \mathrm{Ext}^{3,*}_{BP_*BP}(BP_*, BP_*)$$

is zero (more on γ_1^{alg} very soon—stay tuned), and γ_3^{alg} does not survive the ANSS at the prime 5 because it is the source of a nontrivial differential on the 33rd page. However, things once again stabilize if the prime is large enough.

Key Idea 2.4. (1) *If $p \geq 7$, each member of the gamma family at p yields a non-trivial element in π_*S of order p.*
(2) *The behavior of the Greek letter families we've discussed becomes more regular as the prime increases because of sparseness. The larger the prime, the more spread out the elements of the ANSS are, which in turn means less action with the differentials and more predictable results.*

The element γ_1 (say, at any $p \geq 7$ to be safe) may go down in mathematical history as the biggest troublemaker in the entire apartment buliding. Why? Because γ_1 caused (intentionally, no doubt) significant confusion in the homotopy theory community in the early 1970s. At that time, Shichiro Oka and Hiroshi Toda announced that $\gamma_1 \in \pi_*S$ is zero at a conference in Japan. Also at that time, Emery Thomas and Raphael Zahler announced that $\gamma_1 \neq 0$ in a paper in the Journal of Pure and Applied Algebra [**TZ74**]. Outstanding mathematicians, the four of them, and γ_1 remained elusive, like a celebrity outsmarting the paparazzi. And if that wasn't bad enough, journalists from *Science* [**Sci**] and *The New York Times* [**NYT**] took this snafu in homotopy theory as an opportunity to declare that the decline of mathematics was inevitable! (Slow news cycle?)

Eventually, with the help of Frank Adams, the situation was sorted out and γ_1 is indeed nontrivial in homotopy for $p \geq 7$. But there are still way more questions than there are answers on floor 3. For example, what number-theoretic objects (if any) do the gamma family elements naturally pair up with? And then there are the higher floors, too. What is true about the deltas on floor 4? The epsilons on floor 5? The zetas on floor 6? I don't believe I've ever seen the latter two families even mentioned in print, though I'm sure my colleagues will correct me if I'm wrong. Oh, and what are their number-theoretic counterparts, by the way? And what happens when Greek alphabet is exhausted? Well, almost nothing is known about these higher families or about the $K(n)$-local sphere for $n \geq 4$, whether at small primes or large ones. Nonetheless, homotopy theorists work around the finiteness of the Greek alphabet by letting $\alpha^{(n)}$ denote the n-th Greek letter, so we're covered either way.

A glimmer of hope appears in a recent manuscript by Behrens and Lawson [**BL10**] in which they construct *topological automorphic forms* (or TAF for short), a higher analog of topological modular forms. According to their theory, gaining access to the higher floors of the building requires replacing elliptic curves with higher-genus objects called *Shimura varieties*; it requires replacing modular forms with more general objects called *automorphic forms*; and analogs of $Q(2)$ are conjectured to exist as well. The amount of information required to just *define* TAF is daunting, so while progress in this direction is possible and exciting to think about, it is likely to be slow.

What do you suppose is happening, geometrically or algebraically or in any other respect, with the omicron family (floor 15) at the prime 691? Or, how about the 25th floor, where the Greek letters run out? Or the seven millionth floor? The mind reels.

3. Story of the development

To help myself write about the development of my paper, I dusted off my research notebook from graduate school. It had been a few years since I last looked at it (shame on me). Reading it now brings back a flood of memories. The first few pages call to mind both the excitement I felt starting on my thesis project, and the frustration I felt while being stuck in the mud on numerous occasions. The last few pages, in which my computation finally began to come together, chronicle the sense of relief I felt from knowing I was near the finish line. Something else I noticed—immaterial to anyone besides me, but I'll mention it anyway—is that I used to be a much better note-taker. I used complete sentences, had neat handwriting, and employed different colors of ink for contrast. These days, I write my notes in chicken scratches with a dull pencil that's been thoroughly chewed by my kids. When I'm done writing this I'm buying some new pens. But I digress.

My old research notes reveal that the development of my paper occurred roughly in three stages. In stage 1, I studied the number-theoretic properties of Q-spectra at primes greater than 3 and tried to prove their 3-primary analogs, but failed. In stage 2, still motivated by number-theoretic considerations but unable to make progress on them directly, I turned my attention to simpler, purely computational questions about the homotopy of $Q(2)$ at the prime 3. Finally, stage 3 began with a key breakthrough that I used to carry out my ANSS computation.

3.1. Stage 1. At the start of Topic 1, I noted that the chromatic convergence theorem of Hopkins and Ravenel implies π_*L_2S is a second-order approximation to $\pi_*S\otimes\mathbb{Z}_3$. The theorem itself is actually broader. It says that if X is a spectrum and L_n is localization at the n-th Johnson-Wilson theory $E(n)$ at the prime p, then there is a tower (called the *chromatic tower*)

$$L_0X \leftarrow L_1X \leftarrow L_2X \leftarrow L_3X \leftarrow \cdots$$

whose homotopy limit is (under mild hypotheses on X) the p-localization of X. In particular, the theorem holds for $X = S$ and $p = 3$, yielding the assertion.

Given a map $L_nX \to L_{n-1}X$ from the chromatic tower, its fiber M_nX is the n-th *monochromatic layer* of X. These monochromatic layers sit atop the chromatic tower, as follows:

$$\begin{array}{ccccccccc} & & M_1X & & M_2X & & M_3X & & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ M_0X = L_0X & \longleftarrow & L_1X & \longleftarrow & L_2X & \longleftarrow & L_3X & \longleftarrow & \cdots \end{array}$$

If $X = S$ then $\pi_* M_n S$ zooms in on the n-th floor residents of the metaphorical homotopy apartment building from Topic 2. Moreover, there is a *chromatic spectral sequence* that stitches all of this data together, and it has the form

$$E_1^{s,t} = \pi_t M_s X \Rightarrow \pi_{t-s} X_{(p)}.$$

Behrens proves his correspondence between modular forms and beta elements at primes $p \geq 5$ (as well as other results) by studying $Q(\ell)$ from two perspectives. On the one hand, he studies the chromatic tower of $Q(\ell)$, which ends at $n = 2$ since $Q(\ell)$ is $E(2)$-local. On the other hand, he studies the semi-cosimplicial group $C(\ell)^\bullet$ obtained by applying the functor π_* to the semi-cosimplicial diagram of spectra underlying $Q(\ell)$. [The bullet in the notation $C(\ell)^\bullet$ is a placeholder for dimenison, so $C(\ell)^0 = \pi_* TMF$, $C(\ell)^1 = \pi_*(TMF_0(2) \vee TMF)$, and $C(\ell)^2 = \pi_*(TMF_0(2))$.] These perspectives turn out to be closely linked. We will now sketch the logical chain of ideas that mediates between the two perspectives and yields Behrens' correspondence.

Key Idea 3.1. *Fix a prime $p \geq 5$ and let i range from 0 to 2.*

1. *Since the spectra $Q(\ell)^i$ are built from variants of TMF, the groups $C(\ell)^i$ comprise modular forms.*
2. *For each i there is a spectral sequence whose input is the cohomology of an arithmetically altered version of $C(\ell)^\bullet$, and whose output is $\pi_* M_i Q(\ell)$.*
3. *By sparseness in the aforementioned spectral sequences, there are isomorphisms linking the cohomological data (i.e., modular forms) to the homotopy of the chromatic layers (see* [**Beh09**]*, Corollary 7.7).*
4. *The maps on modular forms given by multiplication by powers of the Eisenstein series E_{p-1} are inclusions, which can be transported over to the monochromatic layers thanks to the aforementioned isomorphisms, where Behrens can leverage them to prove his correspondence (see* [**Beh09**]*, Theorem 11.3).*

Part of what goes wrong with the modular form $E_{3-1} = E_2$ at the prime 3 is that it is not symmetric with respect to the action of the full modular group $\mathrm{SL}(2, \mathbb{Z})$; rather, it is symmetric only with respect to a certain subgroup of $\mathrm{SL}(2, \mathbb{Z})$ (something I eluded to in the previous section). There may be a way around this difficulty with a clever modular form argument. I tried for many months to find such an argument, but at the time I was (and in fact, still am) unable to do so! Something that would have been natural for me to explore during this period is the possibility of explicitly computing $\pi_* M_i Q(2)$ at $p = 3$ for $i = 0, 1, 2$. Maybe I will put my new pens to good use and try it.

3.2. Stage 2. Mid-way through my thesis research, when I was stuck in the manner I described above, I went to visit Mark Behrens in Boston for two weeks. The visit with Mark helped. Perhaps that is not surprising. But, the *way* in which it helped was unexpected—it shifted my focus toward more fertile ground.

The first thing Mark and I discussed was how one might obtain the 3-primary analog of his correspondence, since that has been at the forefront of my mind. Around that time, Mark had consulted a few modular forms experts and they were not sure how to tweak the argument. Mark himself was also not totally sure, although he had some rough ideas and we talked about them. The notes I have from those discussions are technical and I am still parsing through them to this day. However, as our conversations continued, we focused more and more on pure spectral sequence computations for spectra such as TMF and $Q(2)$, as well as the monochromatic layers $M_i Q(2)$.

The ease and fluidity with which Mark drew and dissected spectral sequence charts made it all seem so natural. I had worked through spectral sequence computations before (awkwardly and with a lot of struggle, like most beginners) but Mark got me hooked on them. My desire to get better at basic computations grew, and I began to put considerations about modular forms off to the side. I thought (rightly so!) that I could always return to the number theory questions down the road. I eventually settled on attacking the ANSS for $Q(2)$ and seeing how far I could get with it.

Behrens' explicit description of the cosimplicial structure underlying $Q(2)$ in [**Beh06**] was all anyone needed to embark on this computation. So, when I returned from Boston, I got to work. According to my old research notebook I initially made a lot of progress. But soon after, the computation became unwieldy in a way that I will describe in the next subsection. I was stuck once again.

3.3. Stage 3. The breakthrough that got me unstuck came thanks to a short note Behrens wrote—for his own use, not necessarily for publication—on how to compute the rational homotopy of $Q(2)$ [**Beh**]. He had shown me this note briefly during my visit but I did not read it carefully at that time. In it, Behrens computes the cohomology of the same three-term complex we discussed in Section 1, but *rationally*, on his way to obtaining the result. Behrens ultimately shows that the rational homotopy of $Q(2)$ is concentrated in degrees -2, -1, and 0, meaning $\pi_* Q(2)$ is mostly torsion, as is the case for $\pi_* L_{K(2)} S$. This is evidence that $Q(2)$ does a great job capturing what is happening with the $K(2)$-local sphere.

The methods Behrens uses are from linear algebra, since all the modules in sight are over a field ($\mathbb{C}$, as happens to be the case in Behrens' exposition). Even though working rationally makes computing kernels and cokernels easier, Behrens has to do a decent amount of work to nail things down. The keys to his success are cleverly filtering the diagram, and making judicious choices of $\mathbb{C}$-bases for B, MF, and their various quotients.

My key insight was to observe that Behrens' filtration and his choices of basis could still be used profitably 3-locally.

Key Idea 3.2. *In the diagram of Hopf algebroids*

$$(B,\Gamma) \Rightarrow (B,B) \oplus (B,\Gamma) \Rrightarrow (B,B)$$

the coface maps are maps of $\mathbb{Z}_{(3)}$-algebras, and are determined by their action on polynomial generators. But the alternating sums of coface maps are $\mathbb{Z}_{(3)}$-module maps only, so one has to compute what they do to monomials, and the formulas become complex. Those formulas remain complex after passing to the cochain complex

$$MF \to B \oplus MF \to B$$

in the DCSS (see Topic 1). Behrens encounters a similar difficulty in his rational computation, and proceeds by first filtering the complex, and then choosing $\mathbb{C}$-bases for the rationalized versions of B and MF that drastically simplify computations within the associated graded. Those choices can be mimicked in the 3-local setting.

In other words, much of the computational work in my paper could be characterized as "pseudo-linear algebra," which in this case means linear algebra without inverting 3.

Let us look at an example. Behrens' filtration applied to the cochain complex from my computation takes the form

$$(MF \to B \oplus MF \to B) \supset (MF \to MF \to 0) \supset (0 \to 0 \to 0).$$

After passing to the associated graded, the data necessary to move forward includes the kernel and cokernel of a map $\psi_d + 1 : B \to B$ where 1 is the identity on B and ψ_d is an algebra homomorphism defined by

$$\begin{aligned} \psi_d : q_2 &\mapsto -2q_2, \\ q_4 &\mapsto q_2^2 - 4q_4. \end{aligned}$$

(On the level of Weierstrass equations, ψ_d records the effect of replacing an elliptic curve C by its quotient C/H where H is an order 2 subgroup; or, equivalently, replacing the degree 2 isogeny $C \to C/H$ by its dual isogeny $C/H \to C$, hence the "d" in the notation.)

In Behrens' rational computation, he notes that

$$B \otimes \mathbb{C} = \mathbb{C}[q_2, q_4, q_4^{-1}, \mu^{-1}]$$

and that the set $\{q_4^i \mu^j q_2^\epsilon : i, j \in \mathbb{Z}, \epsilon = 0, 1\}$ is a $\mathbb{C}$-basis for $B \otimes C$. He then breaks up $B \otimes C$ as a direct sum of 2-dimensional subspaces

$$V_{i,j,\epsilon} = \mathbb{C}\{q_4^i \mu^j q_2^\epsilon, q_4^j \mu^i q_2^\epsilon\}$$

each of which is *invariant* under $\psi_d + 1$. Restricting $\psi_d + 1$ to each $V_{i,j,\epsilon}$ yields a 2×2 matrix. Simple.

In my computation, I follow Behrens' lead by noting that $\{q_4^i \mu^j q_2^\epsilon\}$ (where i, j, and ϵ are as above) is a set of generators for B as a $\mathbb{Z}_{(3)}$-module. For computational convenience, I deviate a bit and form the 2-dimensional direct summands

$$V'_{i,j,\epsilon} = \mathbb{Z}_{(3)}\{s^i t^j q_2^\epsilon, s^j t^i q_2^\epsilon\}$$

where $s = 8q_4$ and $t = \mu/8$. They are invariant subspaces of $\psi_d + 1$ and they allow me to study $\psi_d + 1$ as a sequence of 2×2 matrices. In cases where the 2×2 matrix is clearly nonsingular in the rational computation, it is not always so clear 3-locally. I manage to obtain a basis of eigenvectors and compute the eigenvalues, but all the while I must take care to observe whether the eigenvalues are invertible in $\mathbb{Z}_{(3)}$. In fact, they are often not invertible, in which case determining their 3-divisibility is the name of the game.

4. Colloquial summary

In this summary I will—in the context of my paper, as much as possible—talk about what drew me to algebraic topology and homotopy theory, and why I find them exciting! My somewhat circuitous path through graduate school is a big reason why I became a homotopy theorist, so this will be partly autobiographical.

4.1. West coast. Before moving to western New York to write the dissertation that gave rise to the paper addressed by this user's guide, I spent a few formative years in Los Angeles as a math graduate student at UCLA. Those years were formative for three reasons. First and most importantly, I met my wife in southern California. Second, I learned (often the hard way) how hard it is to pursue mathematics seriously. Third, and most relevant to this discussion, I was exposed to the beautiful ideas of number theory.

UCLA had a thriving number theory group while I was there, as it does now. Soon after I arrived I gravitated toward analytic number theory, which in many ways is far removed from algebraic topology. For example, algebraic topologists eat categories and functors for breakfast every morning, while analytic number theorists seem allergic. Of course, I knew very little math at that point so I was blissfully unaware of such distinctions between mathematical diets. The number theorist at UCLA that I gravitated toward the most taught topics in analytic number theory (e.g., modular and automorphic forms, L-functions) from a historical perspective, because he so admired the pioneers of the subject (Hecke, Maaß, Selberg, Ramanujan, etc.). I learned to admire them too, and I had a wonderful time studying mathematics though this "old school" lens. As time went on, however, certain life events made it tough to get a thesis project off the ground at UCLA. I eventually left the program without my PhD. But number theory had left its mark on me, and I had a faint hope that I was not completely done with it.

4.2. East coast. I became interested in algebraic topology shortly after leaving Los Angeles. This was partly the influence of some papers by Dan Freed I stumbled on that blended homotopy theory, geometry, and physics—a blend I found intriguing (and still do). It was also partly the influence of a lecture by Mike Hopkins at the 25th anniversary of MSRI that I happened to watch online. But mostly, I was just in the market for a fresh mathematical start. While I still appreciated number theory and analysis, I yearned for something completely new. Algebraic topology fit that bill. It was a choice made in order to survive as a fledgling mathematician, just as much as it was a choice made out of mathematical taste.

In 2009, Doug Ravenel allowed me to come to Rochester for the fresh start I needed. It was incredibly good fortune, especially because Ravenel held a learning seminar that first fall semester on topological automorphic forms. I remember sitting in that seminar on the first day and thinking, "wow." I could not believe that automorphic forms, these analytic objects that number theorists coveted and that I had ostensibly left behind at UCLA, suddenly appeared in my new mathematical world of algebraic topology. The seminar was a bridge connecting my past mathematical life to my current one. Everything was both familiar and fresh at the same time. Finally, things seemed to fall into place.

Ravenel's TAF seminar led me to Behrens-Lawson manuscript on TAF, which in turn led me to Behrens' papers. That is how I became interested in the Q-spectra specifically.

4.3. The number 504. My excitement for algebraic topology generally, and for homotopy theory in particular, owes a lot to the number 504. If my interest in homotopy theory were an episode of Sesame Street, it would definitely be brought to you by the number 504. This is because 504 happens to be the size of the 11th stable stem: $\pi_{11}S \cong \mathbb{Z}/504$. In fact, this appearance of 504 is a consequence of Adams' work on the image of J, and it's also number-theoretic, because there is a precise sense in which the 504 we see in the stable stems is the same 504 that we see in the recipe for the Eisenstein series Fourier expansion

$$E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}.$$

Even so, it's not like the sizes of stems 0 through 10 give you any warning that 504 is coming up next. They, in fact, give no warning at all. And if we think about this in a more concrete geometric context, we can reasonably ask why it should be that there are exactly 504 ways to throw the 100-dimensional sphere inside the 89-dimensional sphere up to deformation! Why not 503, or 505, or a more sane quantity like 2 or 3? This particular mystery is what has always drawn me (and no doubt others) to homotopy theory, and I feel lucky to be able to think about such mysteries as part of my job.

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